Complex Analysis

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Organized Results

complied by

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Contents

Chapter I

Constructing $\mathbb{C}$

1 For a general ordered field

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Proposition 1.1 (Adjoining $\sqrt{-1}$ to ordered fields)**.** *Let F be an ordered field. Then the operations*

$$\begin{aligned}(a_1, b_1) + (a_2, b_2) &:= (a_1 + a_2, b_1 + b_2), \text{ and} \\ (a_1, b_1)(a_2, b_2) &:= (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2)\end{aligned}$$

make $F \times F$ into a field with

$$\begin{aligned}\text{zero} &= (0, 0) \\ -(a, b) &= (-a, -b) \\ \text{one} &= (1, 0) \\ (a, b)^{-1} &= \left(\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2}\right) \qquad \text{for nonzero } (a, b).\end{aligned}$$

Further, the following hold:

(i) $F \times F$ can't be ordered compatibly.

(ii) $a \mapsto (a, 0)$ is an embedding of F into $F \times F$.[1]

(iii) $F \times F$ forms a two-dimensional vector space over F with a basis $\{1, i\}$.[2]

[1] This allows an abusive notational identification.

[2] We'll denote $(1, 0)$ by 1 (overloading notation) and $(0, 1)$ by i.

Notation. *We'll abuse notation, letting $a+ib$ stand for $a1+bi=(a,b)$ for $a,b\in F$. The functions $\Re$, $\Im$ are just the projections onto the first and second coordinates. When $F=\mathbb{R}$, we denote $F\times F$ (or any isomorphic field) by $\mathbb{C}$.*

2 Specializing to $\mathbb{C}$

Convention. *For the rest of the document, we will spacialize to $\mathbb{C}$.*

Definition 2.1 (Conjugate, norm, argument, inner-product)**.** For $a,b,\theta\in\mathbb{R}$, we define

$$\overline{a+ib} := a-ib, \text{ and}$$
$$|a+ib| := \sqrt{a^2+b^2}.$$

For $z,w\in\mathbb{C}$, we also define

$$\langle z,w\rangle := z\,\overline{w}.$$

Remark. *Note that we will follow the usual convention, defining*

$$0^\alpha := \begin{cases} 1, & \alpha=0 \\ 0, & \alpha>0 \end{cases} \text{ for } \alpha\in\mathbb{R}.$$

Proposition 2.2 (Immediate properties)**.**

(i) We can express $z\in\mathbb{C}$ as

$$z=\Re(z)+i\,\Im(z).$$

(ii) $\langle\cdot,\cdot\rangle$ makes $\mathbb{C}$ into an inner-product space[3] with $|\cdot|$ being the corresponding norm since

$$z\,\bar{z}=|z|^2.$$

$(a,b)\mapsto a+ib$ defines a norm-preserving[4] $\mathbb{R}$-linear isomorphism (and hence also a homeomorphism) $\mathbb{R}^2\to\mathbb{C}$.

(iii) $|\cdot|$ makes $\mathbb{C}$ into a normed algebra with

$$|z\,w|=|z|\,|w|.$$

[3] This allows to define (the *standard*) topology on $\mathbb{C}$.

[4] Note that the inner-product is *not* preserved.

(iv) $z \mapsto \bar{z}$ *defines a norm-preserving*[5] *field automorphism on* $\mathbb{C}$ *with its inverse being itself, since*

$$\begin{aligned}\overline{z+w} &= \bar{z} + \overline{w} \\ \overline{z\,w} &= \bar{z}\,\overline{w} \\ |\bar{z}| &= |z|, \text{ and} \\ \bar{\bar{z}} &= z.\end{aligned}$$

Further,

$$\begin{aligned}\bar{z} = z \quad &\iff z \text{ is real; and,} \\ \bar{z} = -z &\iff z \text{ is imaginary.}\end{aligned}$$

[5] Again, inner-product is *not* preserved in general.

Chapter II

Differentiability

1 Basic definitions

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Convention. *We'll take Ω, Υ to be open subsets of $\mathbb{C}$.*

Definition 1.1 (The derivative)**.** Let $f\colon \Omega \to \mathbb{C}$ and $c \in \Omega$. Then if existent, we define

$$f'(c) := \lim_{z \to c} \frac{f(z) - f(c)}{z - c}.$$

Definition 1.2 (Entire functions)**.** A function $f\colon \mathbb{C} \to \mathbb{C}$ is called entire iff it is differentiable throughout.

Corollary 1.3. *Constant and identity functions are differentiable throughout with derivatives being* 0 *and* 1 *respectively.*

Proposition 1.4. *Differentiability* $\implies$ *continuity.*

Example 1.5. $z \mapsto \bar{z}$ is continuous everywhere but differentiable nowhere!

Proposition 1.6. *Let $f, g\colon \Omega \to \mathbb{C}$ be differentiable at $c \in \Omega$. Then the following hold:*

(i) $(f + g)'(c) = f'(c) + g'(c)$.

(ii) $(\alpha f)'(c) = \alpha f'(c)$ *for* $\alpha \in \mathbb{C}$.

(iii) $(fg)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$.

(iv) $(f/g)'(c) = \big(f'(c)\,g(c) - f(c)\,g'(c)\big)/g(c)^2$ *if* $g(c) \neq 0$.

Corollary 1.7 (Differentiating monomials)**.** *For* $n \geq 1$*, the function* $z \mapsto z^n$ *is differentiable throughout with derivative at* $z \in \mathbb{C}$ *given by* nz^{n-1}*.*

Proposition 1.8 (Chain rule)**.** *Let* $f\colon \Omega \to \Upsilon$ *and* $g\colon \Upsilon \to \mathbb{C}$ *be differentiable at* $c \in \Omega$ *and* $f(c) \in \Upsilon$ *respectively. Then* $g \circ f\colon \Omega \to \mathbb{C}$ *is differentiable at* c *with*

$$(g \circ f)'(c) = g'(f(c))\, f'(c).$$

Convention. *Any bijection* $f\colon X \to X'$ *induces a one-to-one correspondence between functions* $X \to Y$ *and* $X' \to Y$*, and between functions* $Y \to X$ *and* $Y \to X'$ *for a given set* Y*.*

We'll take the bijection $(x, y) \leftrightarrow x + iy$ *while considering* $\mathbb{R}^2$ *and* $\mathbb{C}$*.*

Proposition 1.9 (Making $\mathbb{R}$-linear, $\mathbb{C}$-linear)**.** *The* $\mathbb{C} \to \mathbb{C}$ *map corresponding to the* $\mathbb{R}$*-linear map* $\mathbb{R}^2 \to \mathbb{R}^2$*, given by*

$$x \mapsto Ax, \quad \text{where } A \in \mathbb{R}^{2\times 2},$$

is $\mathbb{C}$*-linear* $\iff$ A *is skew-symmetric.*

Definition 1.10 (Cauchy-Riemann equations)**.** A function $\tilde{f}\colon \tilde{\Omega} \to \mathbb{R}^2$ ($\tilde{\Omega}$ open in $\mathbb{R}^2$) is said to satisfy Cauchy-Riemann equations at $\tilde{c} \in \tilde{\Omega}$ iff in the standard basis, the Jacobian exists and is skew-symmetric, at c, *i.e.*,

$$\begin{aligned} \partial_x \tilde{f}_x(\tilde{c}) &= \partial_y \tilde{f}_y(\tilde{c}), \text{ and} \\ \partial_x \tilde{f}_y(\tilde{c}) &= -\partial_y \tilde{f}_x(\tilde{c}). \end{aligned}$$

Theorem 1.11 (Characterizing differentiability)**.** *Let* $f\colon \Omega \to \mathbb{C}$*, and* $\tilde{f}\colon \tilde{\Omega} \to \mathbb{R}^2$ *be the corresponding function. Let* $c := a + ib \in \Omega$ *(*$a, b \in \mathbb{R}$*). Then the following are equivalent:*

(i) f *is* $\mathbb{C}$*-differentiable at* c *with*

$$f'(c) = \alpha + i\beta, \quad \text{where } \alpha, \beta \in \mathbb{R}.$$

(ii) $\tilde{f}$ *is Fréchet differentiable with*

$$\big[D\tilde{f}(a,b)\big]_{std} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

Convention. *We'll reserve $\mathscr{D}$ to denote a domain in $\mathbb{C}$.*

Example 1.12. A differentiable function $f\colon \mathscr{D} \to \mathbb{C}$ is constant throughout $\mathscr{D}$ if one of the following holds:

(i) $\Re(f(z))$ or $\Im(f(z))$ is constant.

(ii) $|f(z)|$ is constant.

Definition 1.13 (Higher derivatives)**.** Let $f\colon \Omega \to \mathbb{C}$. Then we inductively define the functions $f^{(n)}$'s as:

(i) $f^{(0)} := f$.

(ii) Having defined $f^{(n)}$, define $f^{(n+1)}\colon \Upsilon \to \mathbb{C}$ where Υ is the interior of the set of all the points in the domain of $f^{(n)}$ where it is differentiable, and

$$f^{(n+1)}(z) := \left(f^{(n)}\right)'(z).$$

We say that f is n times differentiable iff $\operatorname{dom} f^{(n)} = \Omega$.

2 Differentiating power series

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Remark. *The results in this section hold for $\mathbb{R}$ also.*

Theorem 2.1 (Differentiating a power series)**.** *Let the complex power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converge to $f\colon B_R(z_0) \to \mathbb{C}$ where $R \in (0,\infty) \cup \{\infty\}$ is the radius of convergence. Then f is differentiable with*[1]

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-z_0)^{n-1}.$$

Corollary 2.2 (Power series are infinitely differentiable)**.** *Let $R \in (0,\infty) \cup \{\infty\}$ be the radius of convergence of the complex power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$. Then the following hold:*

(i) f is k times differentiable for each $k \geq 0$, with

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n (z-z_0)^{n-k}.$$

[1] It is implicitly implied that the radius of convergence of the right-hand-side is R as well.

(ii) c_n*'s are given by*

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Proposition 2.3 (Power series determined by its values on a sequence)**.** *Let* $R \in (0, \infty) \cup \{\infty\}$ *be the radius of convergence of a complex power series* $\sum_{n=0}^{\infty} c_n (z - z_0)^n$*, and let the series vanish*[2] *on a sequence* $(w_n) \in B_R(z_0)$ *one of whose limit points is* z_0*. Then each* $c_n = 0$*.*

3 The exponential function

February 26, 2023

Definition 3.1 (The exponential function)**.** We define $\mathrm{E} \colon \mathbb{C} \to \mathbb{C}$ via[3]

$$z \mapsto \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Proposition 3.2 (Properties of E)**.**

(i) $\mathrm{E}(1) = 1$*.*

(ii) E *is a group homomorphism on* $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ *as well as on* $\mathbb{R} \to (0, \infty)$*.*

(iii) On $\mathbb{R}$*, we have* $\mathrm{E}(x) \to \infty$ *as as* $x \to \infty$ *and* $\mathrm{E}(x) \to 0$ *as* $x \to -\infty$*.*

(iv) E *is differentiable with* $\mathrm{E}' = \mathrm{E}$*.*

(v) $\mathrm{E}(\overline{z}) = \overline{E(z)}$*.*

Proposition 3.3. *For* $x \in \mathbb{R}$*, we have*

$$\mathrm{E}(x) = e^x.$$

4 The trigonometric functions

February 26, 2023

Definition 4.1 (Cosine and sine)**.** We define $\cos, \sin \colon \mathbb{C} \to \mathbb{C}$ as

$$\cos(z) := \frac{\mathrm{E}(iz) + \mathrm{E}(-iz)}{2}, \text{ and}$$
$$\sin(z) := \frac{\mathrm{E}(iz) - \mathrm{E}(-iz)}{2}.$$

[2] That is, it's zero.

[3] Well-defined since the radius of convergence is ∞.

Proposition 4.2 (Properties of the trigonometric functions)**.**

(i) $\cos 0 = 1$ *and* $\sin 0 = 0$.

(ii) $\cos^2 z + \sin^2 z = 1$.

(iii) $\mathrm{E}(iz) = \cos z + i \sin z$.

(iv) $\cos$ *and* $\sin$ *map* $\mathbb{R}$ *in* $[-1, 1]$.

(v) $\cos$ *and* $\sin$ *are differentiable with* $\cos' = -\sin$ *and* $\sin' = \cos$.

Theorem 4.3. *There exists a smallest positive real zero of* $\cos$.

Definition 4.4 (π)**.** We define π to be the real number so that $\pi/2$ is the smallest positive zero of cos.

Proposition 4.5. $\cos(\pi/2) = 0$ *and* $\sin(\pi/2) = 1$.

Proposition 4.6. *On* $[0, \pi/2]$*, we have that* $\cos$ *is strictly decreasing and that* $\sin$ *is strictly increasing.*

Definition 4.7 (Periods of functions)**.** Let G be an additive abelian group and $f\colon G \to G$. Then a $p \in \mathbb{G}$ is called a period of a function $f\colon G \to G$ iff for each $x \in G$, we have

$$f(x + p) = f(x).$$

Lemma 4.8. *For each* $t \in (0, 2\pi)$*, we have that* $\mathrm{E}(it) \neq 0$.

Corollary 4.9. *For* $t \in \mathbb{R}$*, we have that* $\mathrm{E}(it) = 1 \iff t \in 2\pi\mathbb{Z}$.

Proposition 4.10. *For* $z \in \mathbb{Z}$*, we have that*[4]

$$\mathrm{E}(z) = 1 \iff z \in 2\pi i\mathbb{Z}.$$

Corollary 4.11 (2π is the smallest possible period of E, cos, sin)**.** *For* $p \in \mathbb{Z}$*, the following are equivalent:*

(i) ip *is a period of* E.

(ii) p *is a period of* $\sin$.

(iii) p *is a period of* $\cos$.

(iv) $p \in 2\pi\mathbb{Z}$.

[4] Use the unboundedness of E on $\mathbb{R}$.

Theorem 4.12 ($S^1 \leftrightarrow [0, 2\pi)$)**.** *There exists a one-to-one correspondence:*

$$\begin{array}{ccc} [0, 2\pi) & \longleftrightarrow & \{z \in \mathbb{C} : |z| = 1\} \\ \theta & & \mathrm{E}(i\theta) \end{array}$$

Notation. *We use* $S^1 := \{z \in \mathbb{C} : |z| = 1\}$.

Corollary 4.13. *Let* $x, y \in \mathbb{R}$ *not both be zero. Then there exists a unique* $\theta \in [0, 2\pi)$ *such that*

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \quad \textit{and} \quad \sin\theta = \frac{y}{x^2 + y^2}.$$

Corollary 4.14 (Polar form)**.** *Let* $z \in \mathbb{C}\backslash\{0\}$. *Then there exists a unique* $\theta \in (-\pi, \pi]$ *such that*

$$z = |z|\ \mathrm{E}(i\theta).$$

Notation. *We denote the above* θ *by* $\operatorname{Arg} z$.

Corollary 4.15. $\theta \mapsto \mathrm{E}(i\theta)$ *is a continuous surjective group homomorphism on* $\mathbb{R} \to S^1$.

Corollary 4.16 (n-th roots of unity)**.** *Let* $n \geq 1$. *Then* $\mathrm{E}\big(i(2\pi k/n)\big)$, *for* $0 \leq k < n$ *are* n *distinct (and hence all the)* n*-th roots of unity.*

Proposition 4.17 (Subgroups formed by n-th roots of unity)**.** *Define*

$$\Lambda := \bigcup_{n \geq 1} \{n\text{-}th \ roots \ of \ unity\}.$$

Then the following hold:

(i) $\Lambda \lneq S^1 \lneq \mathbb{C} \setminus \{0\}$.

(ii) Λ *is dense in* S^1.

(iii) Let G *be a compact multiplicative subgroup of* $\mathbb{C}$ *and* $f\colon G \to \mathbb{C} \setminus \{0\}$ *be a continuous group homomorphism. Then* $f(G) \subseteq S^1$.

5 Harmonic functions

February 11, 2023
DO THIS AFTER DOING CLAIRAIUT!

Definition 5.1 (Harmonic functions)**.** A function $u\colon \tilde{\Omega} \to \mathbb{R}$ ($\tilde{\Omega}$ open in $\mathbb{R}^2$) is called harmonic iff it is C^2 (with respect to standard bases) and[5]

$$\partial_{x,x} u + \partial_{y,y} u = 0.$$

Corollary 5.2. *First prove that differentiability implies analyticity!*
The real and imaginary parts of a differentiable function on an open set are harmonic.

Definition 5.3 (Harmonic conjugate)**.** Let $u\colon \tilde{\Omega} \to \mathbb{R}$ (Ω open in $\mathbb{R}^2$) be harmonic. Then a harmonic conjugate of u is a harmonic function $v\colon \tilde{\Omega} \to \mathbb{R}$ such that the

[5] We're using a looser notation, not using $\partial_{x,x} u_{(1)}$, *etc.*

Chapter III

Line integrals

1 General things in normed linear spaces

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Convention. *V will stand for a generic normed linear space over $\mathbb{R}$.*[1] *$\mathcal{B}$ will stand for a generic basis for V.*

Convention. *Whenever we'll write $[a, b]$, it will be understood that $a < b$.*

1.1 The derivative of a curve

February 27, 2023

Definition 1.1 (The derivative of a curve)**.** Let $\gamma\colon [a, b] \to V$ be differentiable at $c \in (a, b)$.[2] Then we define[3]

$$\begin{aligned} \gamma'(c) &:= D\gamma(c)(1) \\ &= D_1\gamma(c). \end{aligned}$$

Remark. *This extends the notation for the case when $V = \mathbb{R}$.*

[1]Since a curve is a function $[a, b] \to V$, we better have V over $\mathbb{R}$ in order to take Fréchet (or directional) derivatives.

[2]Since Fréchet differentiability is defined only for open domains.

[3]Since the domain is a subset of $\mathbb{R}$, the directional differentiability is equivalent to Fréchet differentiability.

Corollary 1.2 (Chain rule for curves)**.** *Let $\phi\colon [a,b] \to [c,d]$ be (continuously) differentiable at $x_0 \in (a,b)$ and $\gamma\colon [c,d] \to V$ be (continuously) differentiable at $\phi(x_0) \in (c,d)$. Then $\gamma \circ \phi\colon [a,b] \to V$ is (continuously) differentiable at x_0 with*[4]

$$(\gamma \circ \phi)'(x_0) = \phi'(x_0)\, \gamma'(\phi(x_0)).$$

Corollary 1.3. *Let V be finite-dimensional. Then $\gamma\colon [a,b] \to V$ is differentiable at $c \in (a,b) \iff$ each $\gamma_i\colon [a,b] \to \mathbb{R}$ is differentiable at c, in which case,*

$$\big[\gamma'(c)\big]_{\mathcal{B}} = \begin{bmatrix} \gamma_1'(c) \\ \vdots \\ \gamma_n'(c) \end{bmatrix}.$$

Corollary 1.4 (C^1-ness for curves)**.** *Let V be finite-dimensional. Then for $\gamma\colon [a,b] \to V$, the following are equivalent:*[5]

(i) $\gamma'\colon (a,b) \to V$ is continuous.

(ii) γ is continuously differentiable on (a,b).

(iii) Each $\gamma_i'\colon (a,b) \to \mathbb{R}$ is continuous.

1.2 Operations on curves

February 27, 2023

Definition 1.5 (Negation and join of curves)**.** Let $\gamma\colon [a,b] \to X$ and $\delta\colon [c,d] \to X$ for a set X. Then we define the following:

(i) $-\gamma\colon [a,b] \to X$ given by

$$t \mapsto a + b - t.$$

(ii) $\gamma * \delta\colon [a, b + d - c] \to X$ given by

$$t \mapsto \begin{cases} \gamma(t), & t \in [a,b] \\ \delta(t - b + c), & t \in (b, b + d - c] \end{cases}.$$

Corollary 1.6.

(i) Double negation gives back the original curve.

(ii) Join is associative.

[4]Note that ϕ' is a scalar and $\gamma' \circ \phi$ a vector.

[5]Implicitly is being said in (i) and (iii), that γ and γ_i''s are differentiable in (a,b).

Corollary 1.7. *Continuing Definition 1.5, the following hold:*

(i) *If X is a topological space, then the following hold:*[6]

  (a) *$-\gamma$ preserves the continuity of γ.*

  (b) *$\gamma * \delta$ preserves the continuity of γ and δ on $[a, b)$ and $(c, d]$ respectively. It is continuous at $b \iff \gamma(b) = \delta(c)$, and γ and δ are continuous at b and c respectively.*

(ii) *If $X = V$, then we further have the following:*

  (a) *$-\gamma$ preserves the (continuous) differentiability of γ.*

  (b) *$\gamma * \delta$ preserves the (continuous) differentiability of γ and δ.*[7]

1.3 Integration of curves

February 27, 2023

Proposition 1.8 (Integrating a curve)**.** *Let V be finite-dimensional and $\gamma\colon [a, b] \to V$ be such that each $\gamma_e\colon [a, b] \to \mathbb{R}$ is Riemann-integrable. Then for any other basis $\mathcal{C}$, we also have that each $\gamma_{\tilde{e}}$ is Riemann-integrable with*

$$\sum_{e\in\mathcal{B}} \left(\int_a^b \gamma_e\right) e = \sum_{\tilde{e}\in\mathcal{C}} \left(\int_a^b \gamma_{\tilde{e}}\right) \tilde{e}.$$

Notation. *This allows to denote the above integral by $\int_a^b \gamma$ and also define Riemann-integrability for curves as in Definition 1.9.*

Definition 1.9 (Riemann-integrability for curves)**.** Let $\gamma\colon [a, b] \to V$ with V being finite-dimensional. Then γ is called Riemann-integrable iff there exists a basis of V in which each induced component function $[a, b] \to \mathbb{R}$ is Riemann-integrable.

Remark. *Note that we define $\int_a^b \gamma$ only for finite-dimensional V's.*

Proposition 1.10 (Properties of integrating a curve)**.** *Let V be finite-dimensional and $\gamma, \gamma_1, \gamma_2\colon [a, b] \to V$, and $\delta\colon [c, d] \to V$ be Riemann-integrable. Then the following hold:*[8]

$$\int_a^b (\gamma_1 + \gamma_2) = \int_a^b \gamma_1 + \int_a^b \gamma_2$$

[6] In the following, the preservation is both ways.

[7] Cf. Corollary 1.2.

[8] Implicitly is being said that the integrand curves on the left-hand-side are Riemann-integrable.

$$\int_a^b (\alpha\gamma) = \alpha \int_a^b \gamma \qquad \textit{for any } \alpha \in \mathbb{R}$$
$$\int_a^b (-\gamma) = \int_a^b \gamma \qquad \textit{if } \gamma \textit{ is continuous}$$
$$\int_a^{b+d-c} (\gamma * \delta) = \int_a^b \gamma + \int_c^d \delta \qquad \textit{if } \gamma, \delta \textit{ are continuous}$$

Further, if $V = \mathbb{C}$, then we can also allow $\alpha \in \mathbb{C}$.

Proposition 1.11. *Let V be a finite-dimensional inner-product space (over $\mathbb{R}$) and $\gamma\colon [a, b] \to V$ be Riemann-integrable. Then $\|\gamma\|$ is Riemann-integrable with*

$$\left\| \int_a^b \gamma \right\| \leq \int_a^b \|\gamma\|.$$

Remark. *Next thing would've been an ML formula for inner-product spaces. But we are not defining the line integrals for normed linear spaces...*

Corollary 1.12. *Let V be a finite-dimensional inner-product space (over $\mathbb{R}$). Let $\gamma, \gamma_n\colon [a, b] \to V$ be Riemann-integrable with $\gamma_n \to \gamma$ uniformly.*[9] *Then we have*[10]

$$\int_a^b \gamma_n \to \int_a^b \gamma.$$

Proposition 1.13. *For $\dim V < \infty$, a uniform limit of Riemann-integrable functions is Riemann-integrable.*

Proposition 1.14 ("Integral of derivative")**.** *Let $\dim V < \infty$. Let $\Gamma\colon [a, b] \to V$ be continuous on $[a, b]$ and (Fréchet) differentiable on (a, b). Let $\gamma\colon [a, b] \to V$ be Riemann-integrable with $\gamma = \Gamma'$ on (a, b). Then*[11]

$$\int_a^b \gamma = \Gamma(b) - \Gamma(a).$$

[9] Due to Proposition 1.13, the Riemann-integrability of γ is implied by that of γ_n's.
[10] Corollary 2.8 is the analogue of this result for the *line integrals* in $\mathbb{C}$.
[11] Proposition 2.6 is the analogue of this result for the *line integrals* in $\mathbb{C}$.

1.4 Reparametrizations of curves

February 27, 2023

Definition 1.15 (C^1 parameter transformations)**.** A bijection $\phi\colon [a,b] \to [c,d]$ is called a c.p.t. iff ϕ and ϕ^{-1} are C^1.

Remark. *Equivalently, we could have demanded that ϕ be C^1 with its derivative never vanishing.*

Corollary 1.16. *The derivative of a c.p.t. never vanishes.*

Definition 1.17 (Orientation-preserving and -reversing C^1 parameter transformations)**.** A c.p.t. $\phi\colon [a,b] \to [c,d]$ is called an o.p.c.p.t. iff $\phi' > 0$.[12]

We similarly define o.r.c.p.t.'s.

Proposition 1.18. *The compositions and inverses of c.p.t.'s (respectively o.p.c.p.t.'s) are c.p.t.'s (respectively o.p.c.p.t.'s).*

Definition 1.19 (C^1 reparametrizations)**.** We say that $\delta\colon [c,d] \to V$ is a c.r. of $\gamma\colon [a,b] \to V$ iff $\delta = \gamma \circ \phi$ for some c.r. $\phi\colon [c,d] \to [a,b]$.

Definition 1.20 (Orientation-preserving and -reversing C^1 reparametrizations)**.** We say that $\delta\colon [c,d] \to V$ is an o.p.c.r of $\gamma\colon [a,b] \to V$ iff $\delta = \gamma \circ \phi$ for some o.p.c.p.t. $\phi\colon [c,d] \to [a,b]$.

We similarly define o.r.c.r.'s.

Corollary 1.21. *"Being c.r." (respectively "being o.p.c.r") is an equivalence relation.*

1.5 Nice paths for line integrals

February 28, 2023

Definition 1.22 (Nice paths and induced partitions)**.** A continuous function $\gamma\colon [a,b] \to V$ will be called a *nice path* (or a *nice curve*) iff there are only finitely many points in (a,b) on which γ is not differentiable or is not continuously differentiable, and iff $\|\gamma'\|$ is bounded (wherever γ' exists on (a,b)).

This induces a unique smallest partition $a = x_0 < \cdots < x_n = b$ with $n \geq 1$ such that γ is continuously differentiable in each (x_i, x_{i+1}).[13] We will call this *the partition of* $[a,b]$ *induced by* γ.

[12] The sign of ϕ' at one point determines its sign for the entire interval.

[13] The boundedness of γ' ensure the Riemann-integrability of $\|\gamma'\|(f\circ\gamma)$ in each of the $[x_i, x_{i+1}]$'s.

Remark. *We won't develop the theory of line integrals in normed linear spaces here.*

Proposition 1.23 (O.p.c.r.'s preserve nice-ness)**.** *Let $\gamma\colon [a,b] \to V$ be nice, inducing the partition $a = x_0 < \cdots < x_n = b$. Let $\phi\colon [c,d] \to [a,b]$ be a c.p.t.. Then $\gamma \circ \phi\colon [c,d] \to V$ is also nice, with the induced partition being*

(i) $c = \phi^{-1}(x_0) < \cdots < \phi^{-1}(x_n) = d$ if ϕ is orientation-preserving; and,

(ii) $c = \phi^{-1}(x_n) < \cdots < \phi^{-1}(x_0) = d$ if ϕ is orientation-reversing.

Corollary 1.24 (Negation and joins preserve nice-ness)**.**

(i) Negation preserves the nice-ness of curves.

(ii) Joining nice curves, that match at the joining point, yields a nice curve with the induced partition being "same" except possibly at the joining point.

2 Line integrals in $\mathbb{C}$

February 27, 2023

Remark. *Note that $\mathbb{C}$ is 2-dimensional over $\mathbb{R}$ with $\|\cdot\| = |\cdot|$.*

Definition 2.1 (Line integrals)**.** Let $\gamma\colon [a,b] \to S$ with $S \subseteq \mathbb{C}$ be nice and let $a = x_0 < \cdots < x_n = b$ be the induced partition. Let $f\colon S \to \mathbb{C}$ be continuous on $\gamma([a,b]) \to \mathbb{C}$. Then[14] $(f \circ \gamma)\gamma'$ is Riemann-integrable in each[15] $[x_i, x_{i+1}]$ and we define

$$\int_\gamma f := \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (f \circ \gamma)\gamma'.$$

Remark. *Note that we are defining line integrals only for continuous f's and nice γ's.*

Remark. *In the following, we'll write $\gamma\colon [a,b] \to \mathbb{C}$ and $f\colon \gamma([a,b]) \to \mathbb{C}$.*

[14] Note that the multiplication of $f \circ \gamma$ and γ' is possible since $\mathbb{C}$ is an algebra. For a general normed linear space, only $\|\gamma'\|(f \circ \gamma)$ would've made sense.

Also note that γ' is the Fréchet derivative and not the complex derivative (which doesn't even make sense here).

[15] Note that even though the integrand is not defined at x_i's, we know that the (lower, upper, Riemann) integral are preserved if we change function values at finitely many points.

Example 2.2. Define $\gamma\colon [0,2\pi] \to \mathbb{C}$ by $t \mapsto \mathrm{E}(it)$. Take $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$ given by $z \mapsto z^n$ for $n \in \mathbb{Z}$. Then we have

$$\int_\gamma f = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}.$$

Proposition 2.3 (O.p.c.r.'s preserve line integrals)**.** *Let $\gamma\colon [a,b] \to \mathbb{C}$ be nice and $f\colon \gamma([a,b]) \to \mathbb{C}$ be continuous. Let $\phi\colon [c,d] \to [a,b]$ be an o.p.c.p.t.. Then we have*

$$\int_\gamma f = \int_{\gamma\circ\phi} f.$$

Proposition 2.4 (Properties of line integrals)**.** *Let the curves $\gamma\colon [a,b] \to \mathbb{C}$ and $\delta\colon [c,d] \to \mathbb{C}$ be nice, and match at endpoints. Let $f,g\colon \gamma([a,b]) \to \mathbb{C}$ and $h\colon (\gamma * \delta)([a,b+d-c]) \to \mathbb{C}$ be continuous. Then the following hold:*

$$\int_\gamma (f+g) = \int_\gamma f + \int_\gamma g$$

$$\int_\gamma (\alpha f) = \alpha \int_\gamma f \qquad \text{for any } \alpha \in \mathbb{C}$$

$$\int_{-\gamma} f = -\int_\gamma f$$

$$\int_{\gamma*\delta} h = \int_\gamma h + \int_\delta h$$

Proposition 2.5 (A version of "change of variables")**.** *Let $\gamma\colon [a,b] \to \mathbb{C}$ be nice and $f\colon \gamma([a,b]) \to \mathbb{C}$ be continuous. Let $a,b \in \mathbb{C}$ with $a \neq 0$, and $\phi\colon \mathbb{C} \to \mathbb{C}$ be given by $z \mapsto az+b$. Then $\gamma\circ\phi$ is nice and $f\circ\phi^{-1}$ continuous as well, and we have*

$$a\int_\gamma f = \int_{\phi\circ\gamma} f\circ\phi^{-1}.$$

Proposition 2.6. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be Riemann-integrable. Then $|\gamma|$ is Riemann-integrable as well with*

$$\left|\int_a^b \gamma\right| \leq \int_b^b |\gamma|.$$

Theorem 2.7 (ML formula)**.** *Let $\gamma\colon [a,b] \to \mathbb{C}$ be nice and $f\colon \gamma([a,b]) \to \mathbb{C}$ be continuous. Then*

$$\left|\int_\gamma f\right| \leq \left(\sup_{a\leq t\leq b} |f(\gamma(t))|\right) \underbrace{\sum_i \int_{x_i}^{x_{i+1}} |\gamma'|}_{\text{length of } \gamma}.$$

Corollary 2.8. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be nice and $f, f_n\colon \gamma([a,b]) \to \mathbb{C}$ be continuous such that $f_n \to f$ uniformly on $\gamma([a,b])$. Then*

$$\int_\gamma f_n \to \int_\gamma f.$$

Theorem 2.9 (Integral of derivative)**.** *Let $\gamma\colon [a,b] \to \mathbb{C}$ be nice and $F\colon \Omega \to \mathbb{C}$ be differentiable on $\gamma([a,b]) \subseteq \Omega$. Let $f\colon \gamma([a,b]) \to \mathbb{C}$ be continuous with $f = F'$ on $\gamma([a,b])$. Then*

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)).$$

Corollary 2.10. *The function $z \mapsto 1/z$ doesn't have any "antiderivative" on $\mathbb{C} \setminus \{0\} \to \mathbb{C}$.*

3 Rectangles, Morera, Cauchy

April 8, 2023

Definition 3.1 (Open rectangles and integrals along them)**.** The (complex sets corresponding to[16] the) basic open sets of $\mathbb{R}^2$ will be called open rectangles.

Let R be a nonempty open rectangle and $f\colon S \to \mathbb{C}$ with $S \supseteq \partial R$. Let f be continuous on $\partial R \to \mathbb{C}$. Then we define

$$\oint_{\partial R} f := \int_\gamma f$$

where γ is the nice closed curve given by

$$\gamma := \gamma_1 * \gamma_2 * \gamma_3 * \gamma_4$$

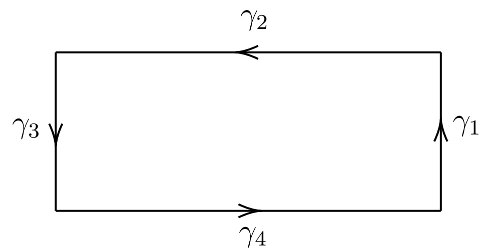
where γ_i's are defined like so:

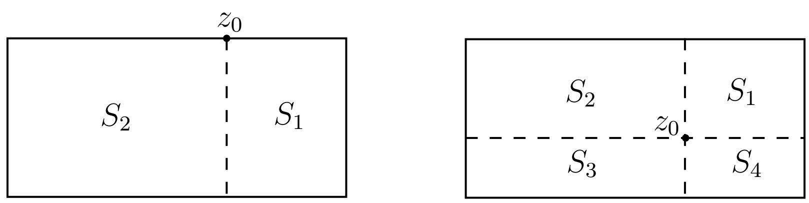
γ_i's have domains $[0,1]$, and are linear, beginning and ending at vertices.

Lemma 3.2. *Let R be a nonempty open rectangle. Let $z_0 \in \overline{R}$ be a "non-corner" point.*[17] *Let $S_1, \ldots, S_n$ be the subrectangles formed as shown, depending whether $z_0 \in \partial R$ or $z_0 \in \operatorname{int}(R)$, respectively:*
Let $f\colon \overline{R} \to \mathbb{C}$ be continuous. Then we have

$$\oint_{\partial R} f = \sum_{i=1}^{n} \oint_{\partial R} f.$$

[16] Note that $(x, y) \to x + iy$ is a homeomorphism.
[17] The "corner points" are not interesting.





Theorem 3.3 (Rectangle theorem)**.** *Let $f\colon \Omega \to \mathbb{C}$ be differentiable throughout and R be an open rectangle such that $\overline{R} \subseteq \Omega$. Then*

$$\oint_{\partial R} f = 0.$$

Further, for a fixed $z_0 \in \Omega$, if we define $g\colon \Omega \to \mathbb{C}$ by

$$g(z) := \begin{cases} \frac{f(z)-f(z_0)}{z-z_0}, & z \neq z_0 \\ f'(z_0), & z = z_0 \end{cases},$$

then also we have[18]

$$\oint_{\partial R} g = 0.$$

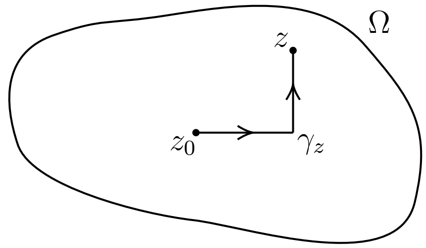
Theorem 3.4 (Morera)**.** *Let Ω be convex and contain a point z_0 such that for every $z \in \Omega$, the shown nice path*[19] *γ_z lies in Ω.*[20]
Let $f\colon \Omega \to \mathbb{C}$ be continuous and such that for every nonempty open rectangle R so that $\overline{R} \subseteq \Omega$, we have

$$\oint_{\partial R} f = 0.$$

The $F\colon \Omega \to \mathbb{C}$ defined by

$$F(z) := \int_{\gamma_z} f.$$

[18] Note that g is continuous throughout, so that we *can* integrate.
[19] Defined similarly as in Definition 3.1.
[20] For instance, open balls and $\mathbb{C}$ are such sets.



is differentiable throughout with

$$F' = f.$$

Remark. *Once Theorem 3.4 holds, we can apply closed curve theorem to f for nice paths lying inside Ω.*

Lemma 3.5. *Let $D_R(z_0) \subseteq \Omega$. Then $B_{R+\varepsilon}(z_0) \subseteq \Omega$ for some $\varepsilon > 0$.*[21]

Remark. *We'll use $\int_\gamma f(z)\, \mathrm{d}z$ notation when the "rule $f(z)$" can be made into a continuous function on (the image of) γ.*

Theorem 3.6 (Cauchy's integral formula)**.** *Let $f\colon \Omega \to \mathbb{C}$ be differentiable throughout. Let $a \in B_R(z_0)$ with $D_R(z_0) \subseteq \Omega$. Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be given by $t \mapsto z_0 + R\,\mathrm{E}(it)$. Then we have*

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, \mathrm{d}z.$$

Corollary 3.7 (Mean value)**.** *Continuing Theorem 3.6, we also have*

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f \circ \gamma.$$

Theorem 3.8 (Differentiability $\implies$ analyticity)**.** *Let $f\colon \Omega \to \mathbb{C}$ be differentiable throughout and $B_R(z_0) \subseteq \Omega$. Then the following hold:*

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[21] *Prove theis! Generalize this!*

(i) f *is infinitely differentiable at* z_0 *with*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

where $\gamma\colon [0, 2\pi] \to \mathbb{C}$ *is any curve given by* $t \mapsto z_0 + r\,\mathrm{E}(it)$*, for some* $0 < r < R$*.*

(ii) *For all* $z \in B_R(z_0)$*, we have a power series representation for* f*:*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Corollary 3.9 (Cauchy's inequality)**.** *Let* $f\colon \Omega \to \mathbb{C}$ *be differentiable at* $z_0 \in \Omega$*. Then for every* R *such that* $B_R(z_0) \subseteq \Omega$*, there exists an* $M > 0$ *such that for each* $n \geq 0$*, we have*

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$

4 Riemann, Liouville, uniqueness

April 9, 2023

Theorem 4.1 (Riemann's removable singularity)**.** *Let* $z_0 \in \Omega$ *and* $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ *be differentiable throughout. Then the following are equivalent:*

(i) f *is differentiably extensible to* Ω*.*

(ii) f *is continuously extensible to* Ω*.*

(iii) f *is bounded around* z_0*.*[22]

(iv) $\lim_{z \to z_0} (z - z_0) f(z) = 0$*.*

Corollary 4.2 (Quotient function of differentiable is differentiable)**.** *Let* $f\colon \Omega \to \mathbb{C}$ *be differentiable at* $z_0 \in \Omega$*. Then the function* $g\colon \Omega \to \mathbb{C}$ *given by*

$$g(z) := \begin{cases} \frac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0 \\ f'(z_0), & z = z_0 \end{cases}$$

is differentiable at z_0 *as well, with the derivative being* $f''(z_0)/2$*.*

[22] That is, in a neighborhood of z_0.

Corollary 4.3. *Let $f\colon \Omega \to \mathbb{C}$ be differentiable throughout and $c_1, \ldots, c_n$ be distinct zeroes of f. Then for the function*

$$g\colon z \mapsto \frac{f(z)}{(z-c_1)\cdots(z-c_n)}$$

defined on $\Omega \setminus \{c_1, \ldots, c_n\}$, each of the limits $\lim_{z\to c_i} g(z)$ exists and the continuous extension of g on Ω is differentiable.

Theorem 4.4 (Liouville)**.** *Let f be entire with*

$$|f(z)| \leq A + B|z|^{\alpha}$$

for some $A, B \geq 0$, $B \neq 0$, $\alpha \in \mathbb{R}$. Then f is a polynomial of degree at most $\max(0, \lfloor \alpha \rfloor)$.

Corollary 4.5. *A nonconstant entire function can't have two $\mathbb{R}$-independent periods.*[23]

Theorem 4.6. *An entire function f such that $|f(z)| \to \infty$ as $|z| \to \infty$ must have zeroes.*

Lemma 4.7. *For any polynomial p, we have that $|p(z)| \to \infty$ as $|z| \to \infty$.*[24]

Corollary 4.8 (The fundamental theorem of algebra)**.** *Any nonconstant complex polynomial must have a root.*

Theorem 4.9 (Uniqueness theorem)**.** *Let $f\colon \mathscr{D} \to \mathbb{C}$ be differentiable throughout. Let f vanish uniformly on $S \subseteq \mathscr{D}$, which has a limit point in $\mathscr{D}$. Then f vanishes everywhere.*[25]

5 Max and min modulus, and open mapping

April 9, 2023

Definition 5.1 (Relative maxima and minima)**.** A point $z_0 \in S \subseteq \mathbb{C}$ is called a relative maximum (respectively minimum) of $f\colon S \to \mathbb{C}$ iff there exists an $\varepsilon > 0$ such that for each $z_\in B_\varepsilon(z_0) \cap S$, we have

$$|f(z_0)| \geq |f(z)| \text{ (respectively } |f(z_0)| \leq |f(z)|).$$

[23]In fact, the such a function in any of the "primitive strips" has to be unbounded!
[24]The usual definition.
[25]We used CC.

Theorem 5.2 (Maximum modulus)**.** *A nonconstant differentiable function on a domain can't have a relative maximum.*

Corollary 5.3 (Minimum modulus)**.** *The relative minima of a nonconstant differentiable function on a domain are precisely its zeroes.*

Remark. *This furnishes another proof of Theorem 4.6.*

Theorem 5.4 (Open mapping theorem)**.** *A differentiable map on an open set is open.*

6 Analytic branches of $\ln(z)$

Do this after doing simply connected regions!

Chapter IV

Laurent series

1 Isolated singularities

April 29, 2023

Definition 1.1. z_0 is called an isolated singularity of a function $f: \Omega \to \mathbb{C}$ iff f is defined, and is differentiable in a deleted neighborhood of z_0.

Example 1.2 (An example of a non-isolated singularity)**.** $z = 0$ for $1/\sin(1/z)$.

Definition 1.3 (Zero of order k)**.** $z_0 \in \mathbb{C}$ is called a zero of order $k \geq 1$ of a function f iff f is differentiable in some $B_R(z_0)$ and if

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is the power series representation of f around z_0, then

(i) $c_0 = \cdots = c_{k-1} = 0$; and,

(ii) $c_k \neq 0$.

Lemma 1.4 (Order of a pole well-defined)**.** *For $j = 1, 2$, let $g_j, h_j: \Omega \to \mathbb{C}$ be differentiable with*

(i) $g_j(z_0) \neq 0$;

(ii) $z_0 \in \Omega$ being a zero of h_j order $k_j \geq 1$;

(iii) h_j*'s vanish only at* z_0*;*[1] *and,*

(iv) $g_j(z)/h_j(z)$*'s coincide on* $\Omega \setminus \{z_0\}$*.*

Then $k_1 = k_2$*.*

Definition 1.5 (Types of isolated singularities)**.** An isolated singularity z_0 of $f\colon \Omega \to \mathbb{C}$ is called

(i) *removable* iff there exists an differentiable function in a neighborhood of z_0 which coincides with f in a deleted neighborhood of z_0;

(ii) *pole of order* $k \geq 1$ iff there exist differentiable functions g, h in a neighborhood of z_0 such that the following hold:

(a) $g(z_0) \neq 0$;

(b) z_0 is a zero of h of order k;

(c) f and g/h coincide in a deleted neighborhood of z_0;

(iii) *essential* iff neither of the above.

Proposition 1.6 (Characterizing poles)**.** *Let* z_0 *be an isolated singularity of* $f\colon \Omega \to \mathbb{C}$ *and* $k \geq 1$*. Then the following are equivalent:*

(i) z_0 *is a pole of* f *of order* k*.*

(ii) $(z - z_0)^k f(z) \not\to 0$ *as* $z \to z_0$ *but* $\lim_{z \to z_0} (z - z_0)^{k+1} = 0$*.*

2 The Laurent expansion

Definition 2.1 (Doubly infinite series)**.** Let $N \in \mathbb{Z}$ and $a_n \in \mathbb{Z}$ for each integer $n \leq N$. Then we define[2]

$$\sum_{n=-\infty}^{N} a_n := \sum_{n=-N}^{\infty} a_{-n}.$$

If $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$, then we define

$$\sum_{n=-\infty}^{\infty} a_n := \sum_{n=-\infty}^{-1} a_n + \sum_{n=0}^{\infty} a_n.$$

Definition 2.2 (Annulus)**.** For $r, R > 0$ and $z_0 \in \mathbb{Z}$, we define

$$A_r^R(z_0) := B_R(z_0) \setminus D_r(z_0).$$

[1] Since $h_j \neq 0$, we can always take a small enough subset of Ω wherein h_j vanishes precisely at z_0.

[2] Of course, the following are defined when the series on the right-hand-side converge.

Corollary 2.3 (Annulus of convergence)**.** *Let the doubly infinite complex series* $\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$ *converge to the function* f *in* $A_r^R(z_0)$*. Then* f *is differentiable with*

$$f'(z) = \sum_{n=-\infty}^{\infty} n c_n (z - z_0)^{n-1}.$$

Remark. *Complex series of the form* $\sum_{n=\infty}^{\infty} c_n (z - z_n)$ *are called Laurent series.*

Theorem 2.4 (Finding the Laurent series)**.** *Let* $f\colon A_r^R(z_0) \to \mathbb{Z}$ *be differentiable. Then* f *admits a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^n + 1} \, \mathrm{d}z.$$

Proposition 2.5 (Characterizing isolated singularities via Laurent series)**.** *Let* z_0 *be an isolated singularity of* $f\colon \Omega \to \mathbb{C}$*. Then* f *admits a Laurent series representation*

$$f(z) = \sum_{n=\infty}^{\infty} c_n (z - z_0)^n$$

in a deleted neighborhood of z_0*, and we have that* z_0 *is*

(i) *a removable singularity* $\iff$ $c_n = 0$ *for all* $n < 0$*;*

(ii) *a pole of order* $k \geq$ $\iff$ $c_{-1} = \cdots = c_{-(k-1)} = 0$ *but* $c_{-k} \neq 0$*; and,*

(iii) *an essential singularity* $\iff$ $c_n \neq 0$ *for infinitely many* $n < 0$*.*

3 Residues

April 29, 2023

Definition 3.1 (Residue around an isolated singularity)**.** Let $f\colon \Omega \to \mathbb{C}$ have an isolated singularity at z_0, and consequently have a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

in a deleted neighborhood of z_0. Then we define

$$\operatorname{Res}(f; z_0) := a_{-1}.$$

Proposition 3.2 (Residue at poles)**.** *Let z_0 be a pole of order $k \geq 1$ for $f \colon \Omega \to \mathbb{C}$. Then we have*

$$\operatorname{Res}(f; z_0) = \frac{1}{(k-1)!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\bigg|_{z=z_0} \big((z - z_0)^k f(z)\big).$$