Contents

In every other result, some form of choice (AC, DC, CC) is used, and we won't bother about enunciating those instances.

Chapter I

Set algebras and additive functions

1 Algebras of sets

August 28, 2023

Definition 1.1 (Closure under finite and countable unions)**.** A collection $\mathcal{E}$ of subsets of a set X is said to be closed under finite (countable) unions iff for each finite (countable) subset $\mathcal{C} \subseteq \mathcal{E}$, we have $\bigcup \mathcal{C} \in \mathcal{E}$.

Lemma 1.2 (Finite and countable unions)**.** *Let X be a set and $\mathcal{E} \subseteq 2^X$. Then the following hold:*

(i) $\mathcal{E}$ is closed under finite unions $\iff$ whenever $A_1, \ldots, A_n \in \mathcal{E}$, we have $\bigcup_{i=1}^n A_i \in \mathcal{E}$.

(ii) $\mathcal{E}$ is closed under countable unions $\iff$ $\emptyset \in \mathcal{E}$ and whenever $A_1, A_2, \ldots \in \mathcal{E}$, we have $\bigcup_{i=1}^\infty A_i \in \mathcal{E}$.

1.1 Algebras and σ-algebras

Definition 1.3 (Algebras and σ-algebras)**.** An algebra (σ-algebra) on a set X is a subset of 2^X that is closed under

(i) finite (countable) unions, and

(ii) complements.

Corollary 1.4. *Let X be a set. Then the following hold:*

(i) $\{\emptyset, X\}$ and 2^X are respectively the smallest and the largest (σ-)algebras on X.

(ii) Nonempty intersections of (σ-)algebras are (σ-)algebras.

(iii) For any $\mathcal{E} \subseteq 2^X$*, the sets*

$$\operatorname{alg}(\mathcal{E}) := \bigcap\{\textit{algebras on } X \textit{ containing } \mathcal{E}\}$$
$$\sigma(\mathcal{E}) := \bigcap\{\sigma\textit{-algebras on } X \textit{ containing } \mathcal{E}\}$$

are respectively the smallest algebra and the smallest σ*-algebra on* X *containing* $\mathcal{E}$*.*

(iv) An algebra (σ*-algebra) is closed under finite (countable) nonempty intersections.*

Remark. *Strictly speaking, the notations* $\operatorname{alg}(\mathcal{E})$, $\sigma(\mathcal{E})$ *should've incorporated* X*. We also call these the (*σ*-)algebra generated by* $\mathcal{E}$*.*

Definition 1.5 (Borel σ-algebra)**.** If X is a topological space, we define $\mathscr{B}(X)$ to be the σ-algebra generated by the open sets of X.

Lemma 1.6. *If* X *is a second-countable topological space, then for any subbase* $\mathcal{S}$*, we have that*

$$\mathscr{B}(X) = \sigma(\mathcal{S}).$$

Result 1.7. The Borel σ-algebra on $\mathbb{R}$ is generated by each of the following collections:

(i) Bounded open (or closed) intervals.

(ii) Bounded half-open-half-closed (or vice versa) intervals.

(iii) Open (or closed) left (or right) rays.

1.2 Semi-algebras

August 29, 2023

Definition 1.8 (Semi-algebras)**.** A semi-algebra on a set X is a subset $\mathscr{S} \subseteq 2^X$ such that the following hold:

(i) $\emptyset \in \mathscr{S}$.

(ii) $\mathscr{S}$ is closed under pairwise intersections.

(iii) If $E \in \mathscr{S}$, then $X \setminus E$ is a finite disjoint union in $\mathscr{S}$.

Proposition 1.9 (From $\mathscr{S}$ to $\operatorname{alg}(\mathscr{S})$)**.** *If $\mathscr{S}$ is a semi-algebra on a set X, then*

$$\operatorname{alg}(\mathscr{S}) = \{\textit{finite disjoint unions in } \mathscr{S}\}.$$

Proposition 1.10 (Semi-algebras over subsets and finite products)**.**

(i) If $\mathscr{S}$ is a semi-algebra over X and $Y \subseteq X$, then

$$\{S \cap Y : S \in \mathscr{S}\}$$

is a semi-algebra over Y.

(ii) If $\mathscr{S}_i$'s are semi-algebras over X_i's for finitely many i's, then

$$\left\{\prod_i S_i : S_i \in \mathscr{S}_i\right\}$$

is a semi-algebra over $\prod_i X_i$.

Result 1.11 (A semi-algebra that generates $\mathscr{B}(\mathbb{R})$)**.** Over $\mathbb{R}^n$, the Cartesian products of the sets of the following kinds forms a semi-algebra:

$$(-\infty, b], \quad (a, b], \quad (a, +\infty), \qquad \text{for } a, b \in \mathbb{R}.$$

Further, the σ-algebra generated by these on $\mathbb{R}$ is precisely $\mathscr{B}(\mathbb{R})$.[1]

1.3 Monotone classes

August 29, 2023

Definition 1.12 (Monotone classes)**.** A monotone class on a set X is a subset $\mathscr{M} \subseteq 2^X$ that is closed under

(i) unions of increasing sequences of sets, and

(ii) intersections of decreasing sequences of sets.

Corollary 1.13. *The analogue of Corollary 1.4 holds for monotone classes as well.*

Remark. *This allows to talk of the monotone class $\mathscr{M}(\mathcal{E})$ generated by $\mathcal{E}$ over X.*

Theorem 1.14 (Monotone class lemma)**.** *Let $\mathscr{A}$ be an algebra over a set X. Then*

$$\mathscr{M}(\mathscr{A}) = \sigma(\mathscr{A}).$$

[1] In fact, this is also true for $n > 1$. (See Proposition 1.13 and Corollary 1.14 (iii).)

1.4 Miscellaneous

August 29, 2023

Notation. *We'll abbreviate "$\mathscr{A}$ is an algebra on a set X" by "$(X, \mathscr{A})$ is an algebra", etc.*

Lemma 1.15 (Constructing a disjoint family of sets)**.** *Let $(X, \mathscr{A})$ be an algebra and $A_1, A_2, \ldots \in \mathscr{A}$. Then*

$$B_n := A_n \setminus \bigcup_{i<n} A_i, \qquad n = 1, 2, \ldots$$

define the unique disjoint sets in $\mathscr{A}$ such that

$$\bigcup_{i=1}^{n} A_i = \dot{\bigcup_{i=1}^{n}} B_i$$

for each n.

Definition 1.16 ($A_i \uparrow A$ and $A_i \downarrow A$)**.** For a sequence of sets $A_1, A_2, \ldots$, we write

- $A_i \uparrow A$ iff $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \bigcup_{i=1}^{\infty} A_i$; and,
- $A_i \downarrow A$ iff $A_1 \supseteq A_2 \supseteq \cdots$ and $A = \bigcap_{i=1}^{\infty} A_i$.

2 Additive functions on set algebras

August 29, 2023

Conventions. *In this section, unless stated otherwise:*

- *X is a set.*
- *$\mathcal{E} \subseteq 2^X$.*
- *$\mu \colon \mathcal{E} \to \mathbb{R}^*$ with each $\mu(E) \geq 0$.*
- *$\mathscr{A}$ is an algebra on X, and Σ a σ-algebra.*

Lemma 2.1 (Finite (sub)additivity)**.**

(i) The following are equivalent:

(a) For any finite $\mathcal{C} \subseteq \mathcal{E}$ with $\bigcup \mathcal{C} \in \mathcal{E}$, we have

$$\mu\Big(\bigcup \mathcal{C}\Big) \leq \sum_{A \in \mathcal{C}} \mu(A).$$

(b) *For any* $A_1, \ldots, A_n \in \mathcal{E}$ *with* $\bigcup_{i=1}^n A_i \in \mathcal{E}$, *we have*

$$\mu\Big(\bigcup_{i=1}^n A_i\Big) \le \sum_{i=1}^n \mu(A_i).$$

(ii) *If* $\emptyset \in \mathcal{E}$ *implies* $\mu(\emptyset) = 0$,[2] *then the following are equivalent:*

(a) *For any finite* $\mathcal{C} \subseteq \mathcal{E}$ *containing disjoint sets with* $\bigcup \mathcal{C} \in \mathcal{E}$, *we have*

$$\mu\Big(\dot{\bigcup} \mathcal{C}\Big) = \sum_{A \in \mathcal{C}} \mu(A).$$

(b) *For any disjoint* $A_1, \ldots, A_n \in \mathcal{E}$ *with* $\bigcup_{i=1}^n A_i \in \mathcal{E}$, *we have*

$$\mu\Big(\dot{\bigcup_{i=1}^n} A_i\Big) = \sum_{i=1}^n \mu(A_i).$$

Lemma 2.2 (Countable (sub)additivity)**.** *Let* $\emptyset \in \mathcal{E}$ *and* $\mu(\emptyset) = 0$.[3] *Then the following hold:*

(i) *The following are equivalent:*

(a) *For any countable* $\mathcal{C} \subseteq \mathcal{E}$ *with* $\bigcup \mathcal{C} \in \mathcal{E}$, *we have*

$$\mu\Big(\bigcup \mathcal{C}\Big) \le \sum_{A \in \mathcal{C}} \mu(A).$$

(b) *For any* $A_1, A_2, \ldots \in \mathcal{E}$ *with* $\bigcup_{i=1}^\infty A_i \in \mathcal{E}$, *we have*

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) \le \sum_{i=1}^\infty \mu(A_i).$$

(ii) *The following are equivalent:*

(a) *For any countable* $\mathcal{C} \subseteq \mathcal{E}$ *containing disjoint sets with* $\bigcup \mathcal{C} \in \mathcal{E}$, *we have*

$$\mu\Big(\dot{\bigcup} \mathcal{C}\Big) = \sum_{A \in \mathcal{C}} \mu(A).$$

(b) *For any disjoint* $A_1, A_2, \ldots \in \mathcal{E}$ *with* $\bigcup_{i=1}^\infty A_i \in \mathcal{E}$, *we have*

$$\mu\Big(\dot{\bigcup_{i=1}^\infty} A_i\Big) = \sum_{i=1}^\infty \mu(A_i).$$

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[2] Required only for "(a) ⇒ (b)".

[3] (i) requires only for "(b) ⇒ (a)", whereas (ii) requires for both directions.

Remark. *Countable (sub)additivity is also called σ-(sub)additivity.*

Result 2.3. Let $\emptyset \in \mathcal{E}$ and $\mu(\dot{\bigcup}\mathcal{C}) = \sum_{A\in\mathcal{C}} \mu(A)$ for all finite $\mathcal{C} \subseteq \mathcal{E}$ containing disjoint sets such that $\bigcup\mathcal{C} \in \mathcal{E}$. Let there also be an $A \in \mathcal{E} \setminus \{\emptyset\}$ such that $\mu(A) < +\infty$. Then $\mu(\emptyset) = 0$.

Definition 2.4 (Some properties of μ and $\mathcal{E}$)**.**

(i) μ is said to be *countably* or *finitely (sub)additive* iff $\emptyset \in \mathcal{E}$ with $\mu(\emptyset) = 0$, and any of the (equivalent) statements of the appropriate list items of Lemmas 2.1 and 2.2 hold.

(ii) μ is called *monotonic* iff whenever $A \subseteq B$ in $\mathcal{E}$, we have $\mu(A) \leq \mu(B)$.

(iii) μ is said to be *continuous from below* (respectively *above*) *at* $A \in \mathcal{E}$ iff $\mu(A) = \lim_i \mu(A_i)$ for every sequence $(A_i) \in \mathcal{E}$ such that $A_i \uparrow A$ (respectively $A_i \downarrow A$ with some $\mu(A_{i_0}) < +\infty$).
If μ is continuous from below (respectively above) at all $A \in \mathcal{E}$, then it's said to be *continuous from below* (respectively *above*).

(iv) μ is called *finite* iff $\mu(A) < +\infty$ for each $A \in \mathcal{E}$.

(v) A subset of X is called *σ-finite* iff it is a countable union of sets in $\mathcal{E}$ on which μ is finite.

(vi) μ is called *σ-finite* iff $\bigcup\mathcal{E}$ is σ-finite.

Corollary 2.5. *Let μ be monotonic and $\bigcup\mathcal{E} \in \mathcal{E}$. Then μ is finite $\iff$ $\mu(\bigcup\mathcal{E}) < +\infty$.*

Proposition 2.6 (Properties of additive functions on algebras)**.** *For $\mathcal{E} := \mathscr{A}$, we have:*

(i) μ is finitely additive $\implies$ μ is finitely subadditive and monotonic.

(ii) μ is σ-additive $\implies$ μ is σ-subadditive and continuous from below as well as from above.

Lemma 2.7 (When is a finitely additive function σ-additive as well?)**.** *Let μ be finitely additive on $\mathcal{E} := \mathscr{A}$. Then we have*

$$\mu \text{ is finite and continuous from above at } \emptyset$$
$$\Downarrow$$
$$\mu \text{ is continuous from below}$$
$$\Downarrow$$
$$\mu \text{ is } \sigma\text{-additive.}$$

Example 2.8 (Some σ-additive functions)**.** Let $\emptyset \in \mathcal{E}$. Then each of the following functions defined on $\mathcal{E}$ is σ-additive:

(i) *(Constantly $+\infty$)*.

$$A \mapsto \begin{cases} 0, & A = \emptyset \\ +\infty, & A \neq \emptyset \end{cases}$$

(ii) *(Counting)*.

$$A \mapsto \begin{cases} \#(A), & A \text{ is finite} \\ +\infty, & \text{otherwise} \end{cases}$$

(iii) *(Dirac)*. For a fixed $x_0 \in X$, define

$$\delta_{x_0}(A) := \begin{cases} 0, & x_0 \notin A \\ 1, & x_0 \in A \end{cases}.$$

3 Measure spaces

August 29, 2023

Definition 3.1 (Measures, null sets, sets of full measures, "almost everywhere")**.** Let (X, Σ) be a σ-algebra. Then a nonnegative σ-additive function $\mu \colon \Sigma \to \mathbb{R}^*$ is called a *(positive) measure* on X, and (X, Σ, μ) is called a *(positive) measure space*. Moreover,

(i) if $\mu(X) = 1$, we call it a *probability measure*;

(ii) sets of Σ on which μ vanishes are called *null*;

(iii) sets of Σ whose complements are null are said to have *full measure*; and,

(iv) a property $P(x)$ pertaining to the elements x of X is said to hold *for almost all* $x \in X$ or *almost everywhere* iff there exists a null set N such that the property holds for each element of $X \setminus N$.[4]

Corollary 3.2. *Nullity and σ-finiteness are preserved under countable unions, whereas having full measure is preserved under countable (nonempty) intersections.*

Definition 3.3 (Functions equal almost everywhere)**.** Functions $f, g \colon X \to Y$, where X is a measure space, are said to be equal almost everywhere iff $f(x) = g(x)$ for almost all $x \in X$.

[4]Note that the property can hold for some elements of N as well.

Notation. *We'll denote the above fact by writing "$f = g$ almost everywhere".*

Remark. *In the same manner, we have obvious definitions for "$f \leq g$ almost everywhere" (when the common codomain is an ordered set), etc.*

Corollary 3.4. *"Being equal almost everywhere" defines an equivalence relation on the set of all the functions from a measure space to a set.*

3.1 Completion of measure spaces

August 29, 2023

Definition 3.5 (Complete measure spaces)**.** A measure space (X, Σ, μ) is called complete if Σ contains all the subsets of its null sets.

Proposition 3.6 (Completion of a measure space)**.** *Let (X, Σ, μ) be a measure space. Then*

$$\overline{\Sigma} := \{E \cup H : E \in \Sigma \text{ and } H \text{ is a subset of some null set}\}$$

is the σ-algebra generated by $\Sigma \cup \{\text{subsets of null sets}\}$. Further, there is a unique measure $\overline{\mu}$ on $\overline{\Sigma}$ that extends μ. Moreover, the following hold:

(i) $(X, \overline{\Sigma}, \overline{\mu})$ *is complete.*

(ii) For any $E \in \Sigma$ and any subset H of a μ-null set, we have

$$\overline{\mu}(E \cup H) = \mu(E).$$

Corollary 3.7 (Complete spaces contain completions)**.** *Let (X, Σ, μ) be a complete measure space and $\Sigma' \subseteq \Sigma$ be another σ-algebra on X. Then with respect to the induced measure, $\overline{\Sigma'} \subseteq \Sigma$.*

4 Constructing measures

August 29, 2023

Theorem 4.1 (From μ on $\mathscr{S}$ to $\operatorname{alg}(\mathscr{S})$)**.** *Let $(X, \mathscr{S})$ be a semi-algebra and $\mu \colon \mathscr{S} \to \mathbb{R}^*$ be a nonnegative finitely (respectively σ-) additive. Then there exists a unique finitely (respectively σ-) additive function ν on $\operatorname{alg}(\mathscr{S})$ that extends μ.*

Lemma 4.2. *Assume:*

- *$\nu\colon \mathscr{A} \to \mathbb{R}^*$ a nonnegative σ-additive function on an algebra $(X, \mathscr{A})$.*
- *γ_1, γ_2 measures on $\sigma(\mathscr{A})$ that extend ν.*
- *$A \in \mathscr{A}$ with $\nu(A) < +\infty$.*

Then γ_1, γ_2 agree on all $E \cap A$'s with $E \in \sigma(\mathscr{A})$.

Theorem 4.3 (At most one measure that extends from $\mathscr{A}$ to $\sigma(\mathscr{A})$)**.** *Let $\nu\colon \mathscr{A} \to \mathbb{R}^*$ be a nonnegative σ-additive function on an algebra $(X, \mathscr{A})$ that is σ-finite as well. Then there exists at most one measure on $\sigma(\mathscr{A})$ that extends ν.*

Example 4.4 (Necessity of σ-finiteness)**.** If $\mathscr{A}$ has no nonempty finite subsets, but $\sigma(\mathscr{A})$ does (as in the case of $\mathbb{R}$), then the constantly $+\infty$ set-function on $\mathscr{A}$ extends to any multiple of the counting measure on $\sigma(\mathscr{A})$.

4.1 Outer measures

August 29, 2023

Definition 4.5 (Outer measure and measurable sets)**.** An *outer measure* π^* on a set X is a nonnegative σ-subadditive monotonic function $\pi^*\colon 2^X \to \mathbb{R}^*$.

$A \subseteq X$ is called *measurable* (with respect to π^*) iff for each $E \subseteq X$, we have

$$\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \setminus A).$$

Lemma 4.6 (The precursor to σ-additivity of $\pi^*|_{\mathcal{M}}$)**.** *Let π^* be an outer measure on a set X and $A_1, \ldots, A_n \subseteq X$ be measurable and disjoint. Then for any $E \subseteq X$, we have*

$$\pi^*\Big(E \cap \dot{\bigcup_{i=1}^{n}} A_i\Big) = \sum_{i=1}^{n} \pi^*(E \cap A_i).$$

Theorem 4.7 (Measurable sets form a complete measure space)**.** *Let π^* be an outer measure on a set X and let $\mathcal{M} := \{$measurable sets$\}$. Then $(X, \mathcal{M}, \pi^*|_{\mathcal{M}})$ forms a complete measure space.*

Theorem 4.8 (From ν on $\mathscr{A}$ to π^*)**.** *Let X be a set and $\emptyset \in \mathcal{E} \subseteq 2^X$. Let $\nu\colon \mathcal{E} \to [0, +\infty]$ with $\nu(\emptyset) = 0$. Define $\pi^*\colon 2^X \to \mathbb{R}^*$ by*

$$\pi^*(A) := \inf\Big\{\sum_{i=1}^{\infty} \nu(E_i) : E_1, E_2, \ldots \in \mathcal{E} \text{ cover } A\Big\}.$$

Then the following hold:

(i) *π^* is an outer measure on X.*

(ii) *If $(X, \mathcal{E})$ is an algebra, then the following hold:*

(a) *ν is finitely additive $\implies$ $\mathcal{E} \subseteq \{$measurable sets$\}$.*

(b) *ν is σ-subadditive and monotonic $\implies$ π^* extends ν.*[5]

(iii) ("Approximating" measurable sets)**.** *If $\mathcal{E} \subseteq \{$measurable sets$\}$ and is closed under finite unions, then for any measurable set A with $\pi^*(A) < +\infty$ and any $\varepsilon > 0$, there exists an $E \in \mathcal{E}$ such that*

$$\pi^*(E \,\Delta\, A) < \varepsilon.$$

Example 4.9 (Necessity of $\pi^*(A) < +\infty$ while approximating)**.** Take $\mathcal{E} = \mathrm{alg}(\mathscr{S})$ on $X = \mathbb{R}$ where $\mathscr{S}$ is as given in Result 1.11. Then no $E \in \mathcal{E}$ can approximate the measurable $A = \bigcup_{n\in\mathbb{Z}}(n - 1/3, n + 1/3)$, within any finite ε.

Corollary 4.10. *Any nonnegative σ-additive function on an algebra $(X, \mathscr{A})$ that is also σ-finite, extends uniquely to a measure on $\sigma(\mathscr{A})$.*

4.2 Regularity

September 1, 2023

Proposition 4.11 ($\mathcal{E}$ approximates $\Sigma \implies \Sigma \subseteq \overline{\sigma(\mathcal{E})}$)**.** *Let (X, Σ, μ) be a measure space and $\mathcal{E} \subseteq \Sigma$ be such that for any $A \in \Sigma$ and any $\varepsilon > 0$, there exist $F, G \in \mathcal{E}$ such that*[6]

- *$X \setminus F \subseteq A \subseteq E$, and*
- *$\mu(E \cap F) < \varepsilon$.*

Then $\Sigma \subseteq \overline{\sigma(\mathcal{E})}$.[7]

4.3 The Lebesgue measure on $\mathbb{R}$

September 3, 2023

[5] Full algebra-ness is not used in this, just that $\mathcal{E}$ is closed under pairwise intersections.

[6] It's easily seen that this "approximation from both sides" condition is equivalent to demanding "approximation from either side".

[7] The completion is with respect to the induced measure on $\sigma(\mathcal{E}) \subseteq \Sigma$.

Lemma 4.12. *Let* $[a,b] \subseteq \bigcup_{i=1}^{n}(a_i,b_i)$ *with* $a<b$ *as well as each* $a_i<b_i$*. Then*

$$b-a<\sum_{i=1}^{n}(b_i-a_i).$$

Theorem 4.13 (The semi-algebra precursor to Lebesgue)**.** *The following defines a* σ*-additive function* μ *on the semi-algebra* $\mathscr{S}$ *on* $\mathbb{R}$ *described in Result 1.11:*

$$\begin{aligned}(-\infty,a],(b,+\infty) &\mapsto +\infty\\ (a,b] &\mapsto \begin{cases} b-a, & a<b\\ 0, & \textit{otherwise}\end{cases}\end{aligned}$$

Definition 4.14 (The Lebesgue measure)**.** Extend the μ in Theorem 4.13 to ν on $\operatorname{alg}(\mathscr{S})$ according to Theorem 4.1. Extend this ν in turn to π^* on $2^{\mathbb{R}}$ as in Theorem 4.8. Then we denote the measure space $(X,\mathcal{M},\pi^*|_{\mathcal{M}})$ obtained via Theorem 4.7 by $(\mathbb{R},\Lambda,\lambda)$, and call Λ the Lebesgue σ-algebra on $\mathbb{R}$.

Theorem 4.15 (Regularity of Lebesgue measure)**.** *Any Lebesgue measurable set can be approximated by open sets in the sense of Proposition 4.11.*

Corollary 4.16. $\Lambda=\overline{\mathscr{B}(\mathbb{R})}$*, where the completion of* $\mathscr{B}(\mathbb{R})\subseteq\Lambda$ *is taken with respect to the restricted* λ*.*

Prove Fubini!

Chapter II

Measurable functions and integrals

Conventions. *For the rest of the document, unless stated otherwise:*

- *A topological space X will also be considered a measurable space (see Definition 1.1) with the σ-algebra being $\mathscr{B}(X)$. In particular, $\mathbb{R}$ will be considered together with $\mathscr{B}(\mathbb{R})$ and not Λ.*
- *The subset of a measurable space will be considered together with the sub-σ-algebra. (See Definition 1.8.)*
- *Any* measurable *subset E of a (signed) measure space will further be considered together with the inherited (signed) measure μ. We'll denote this inherited measure by "μ on E".*
- *The Cartesian product of measurable spaces will be considered together with the product σ-algebra. (See Definition 1.12.)*
- *The pairwise Cartesian product of two measure spaces will further be considered together with the product measure. (See Proposition 3.4.)*
- $0(\pm\infty) := 0 =: (\pm\infty)0$ *so that multiplication becomes a binary operation on $\mathbb{R}^*$. To distinguish it from the usual multiplication where we leave these undefined, we'll call this the "discontinuous multiplication".*
- *For any $f\colon X \to \mathbb{R}$, we will set $f^* := \iota_{\mathbb{R}\to\mathbb{R}^*} \circ f$.*

1 Measurable spaces and functions

September 3, 2023

Definition 1.1 (Measurable spaces)**.** A set together with a σ-algebra on it is called a *measurable space*, and the sets of the σ-algebra are called the *measurable sets* of that space.

Definition 1.2 (Measurable functions)**.** A function between measurable spaces is called measurable iff the inverse images of measurable sets are measurable.

Proposition 1.3 (Suffice to check on generators)**.** *For a function to be measurable, it's enough to have a collection $\mathcal{E}$ that generates the codomain σ-algebra such that the inverse image of each $E \in \mathcal{E}$ is measurable.*

Definition 1.4 (Indicator functions)**.** For a subset $E \subseteq X$, we define the function $\mathbb{1}_E \colon X \to \mathbb{R}$[1] by

$$x \mapsto \begin{cases} 0, & x \notin E \\ 1, & x \in E \end{cases}.$$

Corollary 1.5.

(i) Identity and constant functions are measurable.

(ii) A subset E of a measurable space is measurable $\iff \mathbb{1}_E$ is measurable.[2]

(iii) Continuous functions are measurable.

(iv) Composition preserves measurability.

Proposition 1.6. *If the domain is a complete measure space, then "being equal almost everywhere" preserves measurability of functions.*

Example 1.7 (Necessity of completeness)**.** Let X be a non-complete measure space with a nonmeasurable subset A of a null set N. Then $\mathbb{1}_A = 0$ almost everywhere but 1_A is not measurable.

1.1 Subspaces of measurable spaces

September 3, 2023

[1] We are taking $\mathbb{R}$ and not $\mathbb{R}^*$ since we want it to be simple. (See Definition 2.1.)

[2] $\mathbb{R}$ is considered with its Borel σ-algebra according to the conventions. This however, is not necessary: "$\Rightarrow$" is true for any arbitrary σ-algebra on $\mathbb{R}$, whereas "$\Leftarrow$" is true for all but those in which 0 and 1 can't be separated via measurable sets.

Definition 1.8 (Subspace of a measurable space)**.** Let X be a measurable space and $Y \subseteq X$. Then the smallest σ-algebra on Y that makes the inclusion $Y \hookrightarrow X$ measurable, is called the induced sub-σ-algebra on Y, and the resulting space is called a measurable subspace of X.

Proposition 1.9 (Explicit description of sub-σ-algebra)**.** *Let X be a measurable space and $Y \subseteq X$. Then the induced sub-σ-algebra on Y is precisely*

$$\{E \cap Y : E \text{ is measurable in } X\}.$$

Further, if Y is measurable in X, then the above is precisely the subsets of Y measurable in X.

Proposition 1.10 (Generator of the sub-σ-algebra)**.** *Let X be a measurable space and $Y \subseteq X$. If $\mathcal{E}$ generates the σ-algebra on X, then the induced sub-σ-algebra on Y is generated by*

$$\{E \cap Y : E \in \mathcal{E}\}.$$

Corollary 1.11.

(i) *"Being a subspace" is transitive.*

(ii) *"Having a full measure" is transitive across measure subspaces.*

(iii) *Subspace of a complete measure space is complete.*

(iv) *Restrictions of measurable functions to subspaces are measurable.*

(v) (Enlarging the codomain doesn't affect measurability)**.** *Let X, Y be measurable spaces and $f\colon X \to Y$. Let $f(X) \subseteq Y_1 \subseteq Y$ and $f_1\colon X \to Y_1$ induced via f be measurable. Then f is measurable.*

(vi) (Behavior with topological subspaces)**.** *The Borel σ-algebra of a topological subspace is precisely the sub-σ-algebra induced from the parent Borel σ-algebra.*

(vii) (Pasting lemma)**.** *Let X, Y be measurable spaces and $f\colon X \to Y$. Let E_i's be countably many measurables in X such that each $f \circ \iota_{E_i \to X}$ is measurable. Then f is measurable.*

1.2 Products of measurable spaces

September 3, 2023

Definition 1.12 (Product of measurable spaces)**.** The product σ-algebra of measurable spaces X_i's is defined to be the smallest σ-algebra on $\prod_i X_i$ that makes each π_i measurable. The resulting space is called the product measurable space of X_i's.

Proposition 1.13 (Generator of the product σ-algebra)**.** *Let X_i's be measurable spaces with $\mathcal{E}_i$ generating the σ-algebra of X_i. Then the product σ-algebra of X_i's is generated by*

$$\bigcup_i \{\pi_i^{-1}(E) : E \in \mathcal{E}_i\}.$$

Further, if X_i's are finitely many[3] *and each X_i can be expressed as a countable union in $\mathcal{E}_i$, then the product σ-algebra is also generated by*

$$\left\{\prod_i E_i : E_i \in \mathcal{E}_i\right\}.$$

Corollary 1.14.

(i) (Universal property of product spaces)**.** *Given measurable spaces X and Y_i's, a function $f\colon X \to \prod_i Y_i$ is measurable $\iff$ each $\pi_i \circ f$ is measurable.*

(ii) *Taking subspaces commutes with taking products.*

(iii) (Behavior with topological products)**.** *The Borel σ-algebra of a topological product contains the product σ-algebra. They coincide if the topological product is second-countable.*[4]

1.3 $\mathbb{R}^*$-valued measurable functions

September 3, 2023

Theorem 1.15 ($\mathbb{R}^*$-valued measurable functions)**.** *The following are measurable:*

(i) *Addition on $(\mathbb{R}^* \times \mathbb{R}^*) \setminus E \to \mathbb{R}^*$ where $E := \{(+\infty, -\infty), (-\infty, +\infty)\}$.*

(ii) *The discontinuous multiplication on $\mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}^*$.*

(iii) *Absolute valuation and negation on $\mathbb{R}^* \to \mathbb{R}^*$.*

(iv) *Reciprocation on $\mathbb{R}^* \setminus \{0\} \to \mathbb{R}^*$.*

(v) $\max, \min$ *on* $(\mathbb{R}^*)^n \to \mathbb{R}^*$.

(vi) $\sup, \inf, \limsup, \liminf$ *on* $(\mathbb{R}^*)^{\mathbb{N}} \to \mathbb{R}^*$.[5]

2 The integral

September 15, 2023

[3] Note that $\{0,1\}^{\mathbb{N}}$ is uncountable.

[4] A sufficient condition to ensure second-countability is that there be countably many spaces, each second-countable.

[5] These all are however discontinuous.

Conventions. *In this section, unless stated otherwise:*

- μ *will denote the measure on the measure space being considered.*
- SM_+ *will denote the set of all nonnegative simple measurables (see Definition 2.1) on the measurable space being considered.*
- $\mathcal{L}_+$ *will denote the set of all nonnegative measurables (with codomain* $\mathbb{R}^*$*) on the measurable space being considered.*

Definition 2.1 (Simple functions)**.** A set theoretic function $f\colon X \to \mathbb{R}$ is said to be simple iff its image is finite.

Corollary 2.2.

(i) *Constants and indicators are simple.*

(ii) *Simple functions form an algebra over* $\mathbb{R}$.[6]

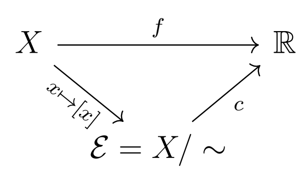
(iii) (Operations that preserve simplicity)**.** *Pointwise addition, multiplication, absolute valuation, negation, reciprocation (whenever defined),* max, min *of simple function(s) are simple.*

Lemma 2.3 (Standard form of simple functions)**.** *Let* $f\colon X \to \mathbb{R}$ *be simple. Then the following hold:*

(i) *There exists a unique finite partition* $\mathcal{E}$ *of* X*, and a unique injection* $c\colon \mathcal{E} \to \mathbb{R}$ *such that*

$$f = \sum_{E \in \mathcal{E}} c_E \mathbb{1}_E.$$

(ii) $\mathcal{E}$ *is the partition of* X *induced by* f *and* c *is the function that makes the following diagram commute:*



(iii) $\operatorname{range} c = \operatorname{range} f$.

(iv) $E = f^{-1}(\{c_E\})$ *for each* $E \in \mathcal{E}$.

(v) *If* X *is a measurable space, then* f *is measurable* $\iff$ *each* $E \in \mathcal{E}$ *is measurable.*

[6] We haven't said anything of this sort for measurable functions yet.

Corollary 2.4. *Let X be a measurable space and $A \in \{(-\infty, 0], \mathbb{R}, [0, +\infty)\}$. Then*

$$\left\{\begin{matrix}\textit{simple measurable}\\ \textit{functions } X \to A\end{matrix}\right\} = \left\{\begin{matrix}A\textit{-linear-combinations of}\\ \textit{indicators of measurables}\end{matrix}\right\}.$$

Remark. *The significance of this lemma is to allow usage of "simple measurables" in place of the cumbersome "linear combinations of indicators of measurables", while at the same time noting that the Borel σ-algebra is not really required to talk of such functions.*

2.1 For nonnegative simple measurables

September 16, 2023

Remark. *We are not allowing positive and negative values both, so as to avoid $(+\infty) + (-\infty)$. And we choose positives over negatives since addition and (discontinuous) multiplication are closed on $[0, +\infty]$.*

Definition 2.5 (Integral of such functions)**.** Let X be a measure space and let f be a nonnegative measurable simple.[7] Then we define[8]

$$\int f \, \mathrm{d}\mu := \sum_{E \in \mathcal{E}} c_E \, \mu(E)$$

where $\mathcal{E}$ and c are as given in Lemma 2.3.

Lemma 2.6 (Relaxing the distinction of coefficients)**.** *Let X be a measure space. Let $\mathcal{E}$ be a finite partition of X into measurables and $c\colon \mathcal{E} \to [0, +\infty)$. Then*

$$\int \Big(\sum_{E \in \mathcal{E}} c_E \mathbb{1}_E \Big) \, \mathrm{d}\mu = \sum_{E \in \mathcal{E}} c_E \, \mu(E).$$

Proposition 2.7 (Properties of $\int$ for such functions)**.** *Let X be a measure space. Then SM_+ is closed under pointwise addition and multiplication. Also, for $f, g \in$*

[7] That is, $f\colon X \to \mathbb{R}$ with each $f(x) \geq 0$.
[8] Note that the sum on the RHS is happening in $[0, +\infty]$.

SM_+, *the following hold:*[9]

$$
\begin{aligned}
\int (f+g)\,\mathrm{d}\mu &= \int f\,\mathrm{d}\mu + \int g\,\mathrm{d}\mu \\
\int (\alpha f)\,\mathrm{d}\mu &= \alpha \int f\,\mathrm{d}\mu && \textit{for } \alpha \in [0,+\infty) \\
\int 0\,\mathrm{d}\mu &= 0 \\
\int f\,\mathrm{d}\mu &\le \int g\,\mathrm{d}\mu && \textit{if } f \le g
\end{aligned}
$$

Further, the function

$$E \mapsto \int (f\,\mathbb{1}_E)\,\mathrm{d}\mu$$

defines another measure on X.

2.2 For nonnegatives

September 27, 2023

Definition 2.8 (Integral of nonnegatives)**.** Let X be a measure space. Then for any nonnegative $f\colon X \to \mathbb{R}^*$,[10] we define

$$\int f\,\mathrm{d}\mu := \sup\Big\{ \textstyle\int g\,\mathrm{d}\mu : g \in \mathrm{SM}_+ \text{ and } g^* \le f \Big\}.$$

Remark. *The notation is being overloaded, but is robust: The codomain ($\mathbb{R}$ or $\mathbb{R}^*$) indicates the case at hand.*

Corollary 2.9.

(i) If f is a nonnegative simple measurable on a measure space X, we have

$$\int f^*\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu.$$

(ii) The integral is monotonic for nonnegatives.

[9] Note that scalar multiplication is just multiplication by a constant function.

[10] That is, each $f(x) \ge 0$.

Theorem 2.10 (Monotone convergence). *Let $f_n, f\colon X \to \mathbb{R}^*$ be nonnegative measurables such that $f_i \uparrow f$. Then*

$$\int f \,\mathrm{d}\mu = \lim_n \int f_n \,\mathrm{d}\mu.$$

Proposition 2.11 (Approximating nonnegative (measurables) by simples). *Let X be a set and $f\colon X \to \mathbb{R}^*$ be nonnegative. For $n \geq 0$, define $f_n\colon X \to \mathbb{R}$ by*[11]

$$f_n(x) := \begin{cases} k/2^n, & f(x) \in k/2^n + [0, 1/2^n) \text{ for}^{12}\ 0 \leq k < n2^n \\ n, & f(x) \in [n, +\infty] \end{cases}.$$

Then the following hold:

(i) f_n*'s are simple nonnegative with* $f_n^* \uparrow f$.

(ii) If f is bounded[13] *on $A \subseteq X$, then $f_n^*|_A \to f|_A$ uniformly*[14] *as well.*

(iii) If X is a measurable space and f measurable, then each f_n^* is measurable.

Proposition 2.12 (Properties of $\int$ for measurable nonnegatives). *Let X be a measure space. Then $\mathcal{L}_+$ is closed under pointwise addition and (discontinuous) multiplication, and for any $f, g \in \mathcal{L}_+$, the following hold:*

(i) $\int (f+g) \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$.

(ii) $\int f \,\mathrm{d}\mu = 0 \iff f = 0$ *almost everywhere.*

(iii) $\int f \,\mathrm{d}\mu < +\infty \implies f^{-1}(\{+\infty\})$ *is null and* $f^{-1}((0, +\infty))$ *is σ-finite.*

(iv) $\int (\alpha f) \,\mathrm{d}\mu = \alpha \int f \,\mathrm{d}\mu$ *for any* $\alpha \in [0, +\infty]$.

(v) $E \mapsto \int (f \mathbb{1}_E^*) \,\mathrm{d}\mu$ *is a measure on X.*

Remark. *Note that although assumption of measurability is crucial for additivity, $\int (f+g) \geq \int f + \int g$ and $\int (\alpha f) = \alpha \int f$ hold for arbitrary nonnegative f, g. So, we are not stating the most refined results.*

Corollary 2.13. *One can interchange $\sum$ and $\int$ for nonnegative measurables.*[15]

[11] f_n's are well-defined since $f \geq 0$.

[12] Thus $x \in \dot\bigcup_{k=0}^{n2^n-1} \big(k/2^n + [0, 1/2^n)\big) = [0, n)$.

[13] Of course, by some finite real.

[14] Note that range $f|_A$ is in the metric space $\mathbb{R}$ so that uniform convergence does make sense.

[15] Of course, defined on a measure space.

2.3 For general integrable functions

September 28, 2023

Definition 2.14 ((General) integrable functions and their integrals)**.** Let X be a measure space. Then $f\colon X \to \mathbb{R}^*$ is said to be a *general integrable function* iff $\int f^+ \,\mathrm{d}\mu$ and $\int f^- \,\mathrm{d}\mu$ are not simultaneously $+\infty$. For such an f, we define

$$\int f \,\mathrm{d}\mu := \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu.$$

A general integrable function which is measurable as well is said to be *integrable*, and an integrable function with a finite integral is said to be *finitely integrable*.

Remark. *This extends Definition 2.8 so that the usage of the same "$\int$" is justified.*

Notation. *When we don't want to introduce a function symbol for a general integrable function $X \to \mathbb{R}^*$ given by $x \mapsto \mathrm{expr}(x)$, we will denote the corresponding integral by*

$$\int \mathrm{expr}(x) \,\mathrm{d}\mu(x).$$

Corollary 2.15. *Nonnegative functions are general integrable, and nonnegative measurables are integrable.*

Theorem 2.16 (Properties of $\int$ for integrable functions)**.** *Let X be a measure space and $f, g\colon X \to \mathbb{R}^*$ be measurable. Then whenever all the terms involved are defined, the following hold:*

(i) $\int (f+g) \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$.

(ii) $\int (\alpha f) \,\mathrm{d}\mu = \alpha \int f \,\mathrm{d}\mu$.

(iii) If $f \le g$, then $\int f \,\mathrm{d}\mu \le \int g \,\mathrm{d}\mu$.

(iv) $\left|\int f \,\mathrm{d}\mu\right| \le \int |f| \,\mathrm{d}\mu$.

(v) $\int |f| \,\mathrm{d}\mu$ *is finite* $\iff$ $\int f^+ \,\mathrm{d}\mu$, $\int f^- \,\mathrm{d}\mu$ *are.*

(vi) If $f = g$ almost everywhere, then $\int f \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$.

(vii) If $\int f \,\mathrm{d}\mu$ is finite, then $f^{-1}(\{-\infty, +\infty\})$ is null and $f^{-1}(\mathbb{R} \setminus \{0\})$ is σ-finite.

Further, the following hold:

(i) If $\int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$ is defined, then $f + g$ is integrable.

(ii) *If* $\int f \, \mathrm{d}\mu$ *is defined and* $\alpha \in \mathbb{R}$*, then* αf *is integrable.*

(iii) *If* $f = g$ *almost everywhere, then* $\int f \, \mathrm{d}\mu$ *is defined* $\iff$ $\int g \, \mathrm{d}\mu$ *is.*

Remark. *Again, we are not stating the most refined results, for some don't require measurability.*

Corollary 2.17. *The set of all finitely integrable functions* $X \to \mathbb{R}^*$ *on a measure space* X *forms a vector space over* $\mathbb{R}$*.*

2.4 Convergence theorems

September 29, 2023

Theorem 2.18 (Generalized monotone convergence)**.** *Let* X *be a measure space and* $f_n, f, g \colon X \to \mathbb{R}^*$ *be integrable such that one of the following groups of assumptions hold:*

(i)
- $f_n \uparrow f$ *almost everywhere.*
- *Each* $f_n \geq g$ *almost everywhere.*
- $\int g \, \mathrm{d}\mu > -\infty$.

(ii)
- $f_n \downarrow f$ *almost everywhere.*
- *Each* $f_n \leq g$ *almost everywhere.*
- $\int g \, \mathrm{d}\mu < +\infty$.

Then

$$\int f \, \mathrm{d}\mu = \lim_n \int f_n \, \mathrm{d}\mu.$$

Theorem 2.19 (Fatou's lemma)**.** *Let* X *be a measure space and* $f_n, f, g \colon X \to \mathbb{R}^*$ *be integrable. Then the following hold:*

(i) *Suppose:*
- $f = \liminf_n f_n$ *almost everywhere.*
- *Each* $f_n \geq g$ *almost everywhere.*
- $\int g \, \mathrm{d}\mu > -\infty$.

Then

$$\int f \, \mathrm{d}\mu \leq \liminf_n \int f_n \, \mathrm{d}\mu.$$

(ii) Suppose:

- $f = \limsup_n f_n$ *almost everywhere.*
- *Each* $f_n \leq g$ *almost everywhere.*
- $\int g \, \mathrm{d}\mu < +\infty$.

Then

$$\int f \, \mathrm{d}\mu \geq \limsup_n \int f_n \, \mathrm{d}\mu.$$

Theorem 2.20 (Dominated convergence)**.** *Let* X *be a measure space and* $f_n, f, g \colon X \to \mathbb{R}^*$ *be integrable such that the following hold:*

- $f_n \to f$ *almost everywhere.*
- *Each* $|f_n| \leq g$ *almost everywhere.*
- $\int g \, \mathrm{d}\mu < +\infty$.

Then[16]

$$\int f \, \mathrm{d}\mu = \lim_n \int f_n \, \mathrm{d}\mu.$$

Proposition 2.21 (Interchanging $\sum$ and $\int$)**.** *Let* X *be a measure space and* $f_n \colon X \to \mathbb{R}^*$ *be integrable such that* $\sum_n \int |f_n| < +\infty$. *Then there exists an integrable function* f *such that* $f = \sum_n f_n$ *almost everywhere*[17] *and*

$$\int f \, \mathrm{d}\mu = \sum_n \int f_n \, \mathrm{d}\mu.$$

3 Product measure and Fubini-Tonelli

3.1 Product of two measure spaces

September 30, 2023

Definition 3.1 (Sections)**.** Let X, Y be sets and $A \subseteq X \times Y$. Then we define

$$A_{x_0} := \{y \in Y : (x_0, y) \in A\} \text{ for } x_0 \in X, \text{ and}$$
$$A^{y_0} := \{x \in X : (x, y_0) \in A\} \text{ for } y_0 \in Y.$$

[16] The existence of the limit on the RHS is also being asserted.

[17] $\sum_n f_n$ is defined on a full-measure set.

Also, if $f\colon X \times Y \to Z$ is any function, we also define functions $f_{x_0}\colon Y \to Z$ and $f^{y_0}\colon X \to Z$ by

$$f_{x_0}(y) := f(x_0, y), \text{ and}$$
$$f^{y_0}(x) := f(x, y_0).$$

Corollary 3.2.

(i) Sections of sets are well-behaved with unions, intersections, deletions and taking subsets.

(ii) Sections of functions are well-behaved under inverse images.

(iii) Section of an indicator function is the indicator of the corresponding section.

Proposition 3.3. *Sections of measurable sets and measurable functions are measurable.*[18]

Proposition 3.4 (Measure on $X \times Y$)**.** *For measure spaces X and Y, there exists a measure μ on $X \times Y$ such that*

$$\mu(E \times F) = \mu_X(E)\,\mu_Y(F)$$

whenever E, F are measurable in X, Y. Such a measure is unique (and σ-finite) if X, Y are σ-finite.

Remark. *The Cartesian product of two measure spaces will always be, unless stated otherwise, considered together with the product measure, as per the conventions.*

3.2 Interchanging integrals

October 6, 2023

Lemma 3.5 (Measuring sections is measurable; Tonelli for indicators)**.** *Let X, Y be σ-finite measure spaces and A be measurable in $X \times Y$. Then the functions*[19]

$$X \to \mathbb{R}^*\colon x \mapsto \mu_Y(A_x), \text{ and}$$
$$Y \to \mathbb{R}^*\colon y \mapsto \mu_X(A^y)$$

are measurable and

$$\int \mu_Y(A_x)\,\mathrm{d}\mu_X(x) = \mu(A) = \int \mu_X(A^y)\,\mathrm{d}\mu_Y(y).$$

[18] We have the product σ-algebra on $X \times Y$ according to the conventions.

[19] These functions are well-defined, thanks to Proposition 3.3

Remark. *Note that* $\mu_Y(A_x) = \int (\mathbb{1}_A)_x \, d\mu_Y$ *and* $\mu_X(A^y) = \int (\mathbb{1}_A)^y \, d\mu_X$.

Theorem 3.6 (Tonelli)**.** *Let* X, Y *be* σ*-finite measure spaces and* $f\colon X \times Y \to \mathbb{R}^*$ *be nonnegative measurable. Then the functions*[20]

$$X \to \mathbb{R}^*\colon x \mapsto \int f_x \, d\mu_Y, \text{ and}$$
$$Y \to \mathbb{R}^*\colon y \mapsto \int f^y \, d\mu_X$$

are nonnegative measurables with

$$\int \left(\int f_x \, d\mu_Y \right) d\mu_X(x) = \int f \, d\mu = \int \left(\int f^y \, d\mu_X \right) d\mu_Y(y).$$

Notation. *The LHS and RHS of the above will also sometimes be denoted by* $\iint f \, d\mu_Y \, d\mu_X$ *and* $\iint f \, d\mu_X \, d\mu_Y$.

Example 3.7 (Necessity of σ-finiteness)**.** Let λ and μ respectively be the Lebesgue and the counting measures on $\mathscr{B}(\mathbb{R})$. Let $E \in \mathscr{B}(\mathbb{R})$ such that $A := \{(x, x) : x \in E\}$ is measurable.[21] Then

$$\iint \mathbb{1}_A \, d\lambda \, d\mu = 0, \text{ but}$$
$$\iint \mathbb{1}_A \, d\mu \, d\lambda = \lambda(E).$$

Theorem 3.8 (Fubini)**.** *Let* X, Y *be* σ*-finite measure spaces and* $f\colon X \times Y \to \mathbb{R}^*$ *be finitely integrable. Let* $g\colon X \to \mathbb{R}^*$ *be given by*[22]

$$g^+(x) = \begin{cases} \int (f^+)_x \, d\mu_Y, & \int (f^+)_x \, d\mu_Y < +\infty \\ 0, & \text{otherwise} \end{cases}, \text{ and}$$
$$g^-(x) = \begin{cases} \int (f^-)_x \, d\mu_Y, & \int (f^-)_x \, d\mu_Y < +\infty \\ 0, & \text{otherwise,} \end{cases}$$

[20] These are well-defined, again due to Proposition 3.3.
[21] For instance, take $E = [0, 1]$.
[22] Note that g, h are well-defined by this description.

and $h\colon Y \to \mathbb{R}^*$ *be given by*

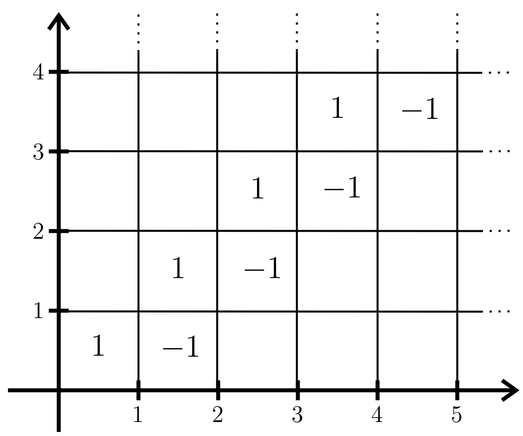
$$h^+(y) = \begin{cases} \int (f^+)^y \,\mathrm{d}\mu_X, & \int (f^+)^y \,\mathrm{d}\mu_Y < +\infty \\ 0, & \textit{otherwise} \end{cases}, \textit{ and}$$
$$h^-(y) = \begin{cases} \int (f^-)^y \,\mathrm{d}\mu_X, & \int (f^-)^y \,\mathrm{d}\mu_X < +\infty \\ 0, & \textit{otherwise.} \end{cases}$$

Then g*,* h *are integrable with*

$$\int g \,\mathrm{d}\mu_X = \int f \,\mathrm{d}\mu = \int h \,\mathrm{d}\mu_Y.$$

Notation. *As before,* $\iint f \,\mathrm{d}\mu_Y \,\mathrm{d}\mu_X$ *and* $\iint f \,\mathrm{d}\mu_X \,\mathrm{d}\mu_Y$ *will also be used to denote the LHS and RHS of the above.*

Example 3.9 (Necessity of finite integrability)**.** Consider the following measurable function $f\colon \mathbb{R}^2 \to \mathbb{R}^*$:[23]



Then

$$\iint f(x,y) \,\mathrm{d}\lambda(x) \,\mathrm{d}\lambda(y) = 0, \text{ however}$$
$$\iint f(x,y) \,\mathrm{d}\lambda(y) \,\mathrm{d}\lambda(x) = 1.$$

[23] f is defined piecewise on the squares $[n, n+1) \times [m, m+1)$.

4 Signed measures

October 8, 2023

Conventions. *In this section, unless stated otherwise:*

- *"μ is a [positive, signed] measure on X" will mean that μ is a [positive, signed] measure on a measurable space X. (See Definition 4.1.)*

Definition 4.1 (Signed measures and null, positive, negative sets)**.** A *signed measure* on a measurable space (X, Σ) is a function $\mu\colon \Sigma \to \mathbb{R}^*$ such that the following hold:

- μ doesn't attain both $-\infty$ and $+\infty$.
- $\mu(\emptyset) = 0$.
- For any disjoint $E_1, E_2, \ldots \in \Sigma$, we have[24]

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i).$$

We call (X, Σ, μ) a *signed measure space*. Further, any measurable $E \subseteq X$ is called *positive* (respectively *negative*, *null*) iff every measurable subset of E is of nonnegative (respectively nonpositive, zero) measure.

Remark. *Positive measures are signed measures and the definition of null sets here extends that in the case of positive measures.*

Corollary 4.2. *Let μ be a signed measure on X. Then the following hold:*

(i) For disjoint measurables $E_1, E_2, \ldots \subseteq X$,
  - *(a) if $\mu(\bigcup_i E_i)$ is finite, then $\sum_i \mu(E_i)$ converges absolutely (in $\mathbb{R}$); and,*
  - *(b) if $\mu(\bigcup_i E_i)$ is infinite, then no rearrangement of $\sum_i \mu(E_i)$ is finite.*

(ii) μ is finitely additive.

(iii) Subsets of finite measure can't have subsets of infinite measure.

(iv) μ is continuous from below and from above.[25]

(v) Positivity, negativity and nullity of measurables is preserved while taking subsets or countable unions.

[24]Note that the sum on the RHS is taking place in either $\mathbb{R} \cup \{-\infty\}$ or $\mathbb{R} \cup \{-\infty\}$.

[25]According to the obvious definitions.

(vi) A positive or a negative set with zero measure is null.

(vii) μ is monotonic on positive and negative measurables.

Example 4.3 (Some signed measures)**.**

(i) Let μ, ν be positive measures on X with one of them being finite. Then $\mu - \nu$ is a signed measure on X.

(ii) Let μ be a positive measure on X and $f\colon X \to \mathbb{R}^*$ be integrable. Then $E \mapsto \int f\, \mathbb{1}_E^* \,\mathrm{d}\mu$ defined a signed measure on X.

4.1 Hahn decomposition

October 8, 2023

Lemma 4.4. *Let μ be a signed measure on X and $E \subseteq E$ be a nonpositive (respectively nonnegative) measurable with $|\mu(E)| < +\infty$. Then E contains a measurable F such that $\mu(F) > \mu(E)$ (respectively $\mu(F) < \mu(E)$).*

Proposition 4.5. *Let μ be a signed measure on X. Then the following hold:*

(i) There exist positive and negative sets having the greatest and respectively the least measures.

(ii) Let μ not attain $+\infty$ (respectively $-\infty$) and let $E \subseteq X$ be a nonnegative (nonpositive) measurable. Then E contains a positive (respectively negative) set of positive (respectively negative) measure.

Corollary 4.6 (Hahn decomposition)**.** *Let μ be a signed measure on X. Then there X is partitioned by a pair of disjoint positive and negative sets.*

Further, if P, N and P', N' are two such decompositions, then for any measurable E, we have

$$\mu(E \cap P) = \mu(E \cap P'), \text{ and}$$
$$\mu(E \cap N) = \mu(E \cap N').$$

In particular, $P \,\Delta\, P'$ and $N \,\Delta\, N'$ are null.

Remark. We'll call such a decomposition a Hahn decomposition of X with respect to μ.

4.2 Jordan decomposition

October 8, 2023

Definition 4.7 (Mutually singular measures)**.** Two signed measures μ, ν on X are called mutually singular, denoted $\mu \perp \nu$ iff X can be partitioned into a μ-null and a ν-null set.

Proposition 4.8 (Jordan decomposition)**.** *Let μ be a signed measure on X and $X = P \mathbin{\dot\cup} N$ be a Hahn decomposition of X. Then*

$$\mu^+(E) := \quad \mu(E \cap P), \text{ and}$$
$$\mu^-(E) := -\mu(E \cap N)$$[26]

define the unique mutually singular positive measures μ^+, μ^- on X that are never simultaneously $+\infty$, with

$$\mu = \mu^+ - \mu^-.$$

Definition 4.9 (Variations of a signed measure)**.** In Proposition 4.8, we call μ^+ and μ^- the *positive* and *negative variations* of μ. We also define the measure

$$|\mu| := \mu^+ + \mu^-$$

to be the *total variation* of μ.

Remark. *Note that unlike functions, $|\mu|(E) \neq |\mu(E)|$. We only have $|\mu(E)| \leq |\mu|(E)$.*

Corollary 4.10. *Let μ, ν be signed measures on X and $E \subseteq X$ be a measurable. Then the following hold:*

(i) μ *is positive* $\iff \mu^- = 0 \iff |\mu| = \mu$.

(ii) E *is* μ*-null* $\iff$ E *is* μ^+*-*, μ^-*-null* $\iff$ E *is* $|\mu|$*-null.*

(iii) $\mu \perp \nu \iff \mu^+, \mu^- \perp \nu \iff |\mu| \perp \nu$.

[26] Had we defined integrals for signed measures, we could've written $\mu^+(E) = \int_E \mathbb{1}_P \,\mathrm{d}\mu$ and $\mu^-(E) = -\int_E \mathbb{1}_N \,\mathrm{d}\mu$.

4.3 Lebesgue-Radon-Nikodým

October 8, 2023

Definition 4.11 ((σ-)finiteness of signed measures)**.** A signed measure is called (σ)-finite iff the its total variation is (σ)-finite.

Definition 4.12 (Absolute continuity)**.** Let μ, ν be signed measures on X. Then ν is said to be absolutely continuous with respect to μ, written $\nu \ll \mu$ iff for every measurable $E \subseteq X$, we have[27]

$$|\mu|(E) = 0 \implies \nu(E) = 0.$$

Corollary 4.13.

(i) *Let μ, ν be signed measures on X. Then $\nu \ll \mu \iff \nu^+, \nu^- \ll \mu \iff |\nu| \ll \mu$.*

(ii) *Let μ, ν be signed measures on X. If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.*

Proposition 4.14 (Why called "absolute continuity"?)**.** *Let μ, ν be signed measures on X with ν finite. Then $\nu \ll \mu \iff$ for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$|\mu|(E) < \delta \implies |\nu(E)| < \varepsilon$$

for all measurables $E \subseteq X$.

Lemma 4.15 (Characterizing mutual singularity)**.** *Let μ, ν be positive measures on X with μ finite. Then $\mu \perp \nu \iff$ no measurable $E \subseteq X$ with $\mu(E) > 0$ is a positive set for $\nu - \varepsilon\mu$[28] for any $\varepsilon > 0$.*

Lemma 4.16. *Let X be a measurable space and $f, g \colon X \to \mathbb{R}^*$ be measurable. Then the set*

$$\{x \in X : f(x) < g(x)\}$$

is measurable.

Theorem 4.17 (Lebesgue-Radon-Nikodým)**.** *Let μ, ν be σ-finite signed measure measures on X with μ positive. Then there exist unique σ-finite signed measures ν_1 and ν_2 on X, and a μ-integrable function $f \colon X \to \mathbb{R}$ unique up to being equal almost everywhere, such that*

(i) *$\nu_1(E) = \int f^* \mathbb{1}_E^* \, \mathrm{d}\mu$ for any measurable E; in particular, $\nu_1 \ll \mu$,*

(ii) *$\nu_2 \perp \mu$, and*

(iii) *$\nu = \nu_1 + \nu_2$.*[29]

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[27] Thus, $\nu \ll \mu \iff \nu \ll |\mu|$.

[28] That is, $\nu \not\geq \varepsilon\mu$ on E.

[29] Implicitly is being stated that the sum is always well-defined.

Further, if ν *is positive, then* ν_1, ν_2 *are positive as well.*

Chapter III

Modes of convergence

1 Basic stuff

October 10, 2023

Lemma 1.1 (Pointwise and uniform convergence descend to equivalence classes)**.** *Let X be a measure space and Y a Hausdorff topological (metric) space. Let f_n, $g_n, f, g\colon X \to Y$ be functions such that $f_n \to f$ (uniformly) almost everywhere and each $f_n = g_n$ almost everywhere. Then $f = g$ almost everywhere $\iff$ $g_n \to g$ (uniformly) almost everywhere.*

2 $\operatorname{ess\,inf}$ and $\operatorname{ess\,sup}$

October 18, 2023

Definition 2.1 ($\operatorname{ess\,sup}$ and $\operatorname{ess\,inf}$)**.** Let X be a measure space and $f\colon X \to \mathbb{R}^*$. Then we define

$$\operatorname{ess\,inf} f := \sup\{\alpha \in \mathbb{R}^* : f \geq \alpha \text{ almost everywhere}\}, \text{ and}$$
$$\operatorname{ess\,sup} f := \inf\{\alpha \in \mathbb{R}^* : f \leq \alpha \text{ almost everywhere}\}.$$

Proposition 2.2 (Properties of $\operatorname{ess\,sup}$ and $\operatorname{ess\,inf}$)**.** *Let X be a measure space and $f, g\colon X \to \mathbb{R}^*$. Then the following hold:*

(i) $\inf f \leq \operatorname{ess\,inf} f$ *and* $\operatorname{ess\,sup} f \leq \sup f$.

(ii) If $\mu(X) > 0$, then $\operatorname{ess\,inf} f \leq \operatorname{ess\,sup} f$.[1]

[1] Otherwise, $\operatorname{ess\,sup} f = -\infty < +\infty = \operatorname{ess\,inf} f$.

(iii) $\operatorname{ess\,inf} f \leq f \leq \operatorname{ess\,inf} f$ *almost everywhere.*

(iv) (Descent to equivalence classes). *Let* $f = g$ *almost everywhere. Then* $\operatorname{ess\,inf} f = \operatorname{ess\,inf} g$ *and* $\operatorname{ess\,sup} f = \operatorname{ess\,sup} g$.

(v) *If* $f \leq g$ *almost everywhere, then* $\operatorname{ess\,inf} f \leq \operatorname{ess\,inf} g$ *and* $\operatorname{ess\,sup} f \leq \operatorname{ess\,sup} g$.

(vi) (ess inf, ess sup as inf, sup). *Define* $\tilde{f}\colon X \to \mathbb{R}^*$ *by*

$$\tilde{f}(x) := \begin{cases} \operatorname{ess\,inf} f, & f(x) \in [-\infty, \operatorname{ess\,inf} f) \\ f(x), & f(x) \in [\operatorname{ess\,inf} f, \operatorname{ess\,sup} f] \\ \operatorname{ess\,sup} f, & f(x) \in (\operatorname{ess\,sup} f, +\infty] \end{cases}.$$

Then the following hold:

(a) $\tilde{f} = f$ *almost everywhere.*

(b) $\operatorname{ess\,inf} f = \inf \tilde{f}$ *and* $\operatorname{ess\,sup} f = \sup \tilde{f}$.

(c) $\tilde{f}$ *is measurable if* f *is.*

3 Uniform metric and uniform convergence

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Definition 3.1 (Distance between functions)**.** Let X be a measure space and f, $g\colon X \to Y$ where Y is a metric space. Then we define the distance between f and g by[2]

$$d(f, g) := \operatorname*{ess\,sup}_{x \in X} d\big(f(x), g(x)\big).$$

Corollary 3.2 (The distance is almost a metric)**.** *Let* X *be a measure space and* Y *a metric space. Then for* $f, g, h\colon X \to Y$*, the following hold:*

(i) $d(f, g) \geq 0$ *if* $\mu(X) > 0$.

(ii) $d(f, g) \leq 0 \iff f = g$ *almost everywhere.*

(iii) $d(f, g) = d(g, f)$.

(iv) $d(f, h) \leq d(f, g) + d(g, h)$.

Proposition 3.3 (Distance and uniform convergence)**.** *Let* X *be a measure space and* Y *a metric space. Then for* $f_n, f\colon X \to Y$*, we have* $f_n \to f$ *uniformly almost everywhere* $\iff d(f_n, f) \to 0$.

[2] Of course, the function being considered here is $x \mapsto d(f(x), g(x))$.

4 Convergence in measure

October 18, 2023

Definition 4.1 (Convergence in measure)**.** Let X be a measure space and Y a metric space. Let $f_n, f\colon X \to Y$ be measurables. Then we say that $f_n \to f$ in measure iff for every $\varepsilon > 0$, we have that

$$\lim_{n\to\infty} \mu(X \setminus E_n) = 0$$

where $E_n := \{x \in X : d(f_n(x), f(x)) < \varepsilon\}$.[3]

Corollary 4.2. *Uniform convergence almost everywhere $\implies$ convergence in measure.*

Proposition 4.3 (Descent to equivalence classes)**.** *Let X be a measure space and Y a metric space. Let $f_n, g_n, f, g\colon X \to Y$ be measurable such that $f_n \to f$ in measure and each $f_n = g_n$ almost everywhere. Then $f = g$ almost everywhere $\iff$ $g_n \to g$ in measure.*

4.1 Relation with pointwise convergence

Example 4.4 (Convergence in measure $\not\Rightarrow$ pointwise convergence)**.** Define f_n's on $\mathbb{R}$ as follows:

$$\begin{aligned} &f_1 := \mathbb{1}_{[0,1]} \\ &f_2 := \mathbb{1}_{[0,1/2]} \quad f_3 := \mathbb{1}_{[1/2,1]} \\ &f_4 := \mathbb{1}_{[1,1/4]} \quad f_5 := \mathbb{1}_{[1/4,1/2]} \quad f_6 := \mathbb{1}_{[1/2,3/4]} \quad f_7 := 1_{[3/4,1]} \\ &\vdots \end{aligned}$$

Then $f_n \to 0$ in measure. However, $f_n(x) \not\to 0$ for any $x \in [0, 1]$.

Proposition 4.5. *Convergence in measure $\implies$ there's a subsequence that converges almost everywhere.*

Example 4.6 (Pointwise convergence $\not\Rightarrow$ convergence in measure)**.** Consider $f_n := \mathbb{1}_[n, +\infty)$ on $\mathbb{R}$. Then $f_n \to 0$ pointwise, but not in measure.

Proposition 4.7. *In a finite measure space, pointwise convergence almost everywhere $\implies$ convergence in measure.*

[3] Each E_n is measurable since f_n, f are.