Real Analysis

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Organized Results

complied by

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To Giuseppe,
for inspiring me once,
and then for all...

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Contents

Chapter I

Real number system

1 Dedekind cuts of an Archimedean ordered field

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Remark. *We'll fix an ordered Archimedean field F in this section, and denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, the appropriate embeddings in F.*

Definition 1.1 (Dedekind cuts)**.** Let $A \subseteq F$. Then A is called a (Dedekind) cut iff the following hold:

(i) $A \neq \emptyset, F$.

(ii) A is "closed downwards", *i.e.*,

$$a \in A \text{ and } a' \leq a \implies a' \in A.$$

(iii) A has no maximum.

Proposition 1.2 (Sum of cuts)**.** *Let A, B be cuts. Then the set*

$$A + B := \{a + b : a \in A, b \in B\}$$

is a cut.

Proposition 1.3. *Sum of cuts is commutative and associative.*

Proposition 1.4 (F's embedding in the set of cuts)**.** *Let $x \in F$. Then the set*

$$x^* := (-\infty, x)$$

is a cut.

Proposition 1.5. *0^* is the additive identity for cuts.*

Lemma 1.6. *Let A be a cut such that $F \setminus A$ has a minimum, namely L. Then*

$$A = (\infty, L).$$

Proposition 1.7 (Negation of cuts)**.** *Let A be a cut. Then the set*

$$-A := \begin{cases} -(F \setminus A), & F \setminus A \text{ has no minimum} \\ -(F \setminus (A \cup \{L\})), & L \text{ is the minimum of } F \setminus A \end{cases}$$

is a cut.

It follows that

$$-A = \{x \in F : \text{for some } y \in F \setminus A, \text{we have } y < -x\}.$$

Remark. We have abused the notation $-X$ above for $X \subseteq F$ due to overloading.

Show the "well-ordering" for $\mathbb{Z}$ inside F.

Proposition 1.8. *Negation of a cut is its additive inverse.*

Remark. Proposition 1.8 is the first time that Archimedean-ness is used.

Theorem 1.9. *Cuts form an abelian additive group.*

Definition 1.10 (Order on cuts)**.** For cuts A and B, we write

$$A \leq B \text{ iff } A \subseteq B.$$

Theorem 1.11. *$\leq$ is a total order for cuts, and it preserves addition, i.e.,*

$$A \leq B \implies A + C \leq B + C.$$

Proposition 1.12 (Product of cuts)**.** *Let A, B be cuts with $A, B \geq 0^*$. Then the set*

$$AB := 0^* \cup \{ab : a \in A \setminus 0^*, b \in B \setminus 0^*\}.$$

forms a cut.

Using this, for any cuts A, B, we define

$$AB := \begin{cases} AB, & A, B \geq 0^* \\ -((-A)B), & A < 0^*, B \geq 0^* \\ -(A(-B)), & A \geq 0^*, B < 0^* \\ (-A)(-B), & A, B < 0^* \end{cases}$$

which is again a cut.

Remark. *Again, we have abused notation slightly. (We must have denoted the first product by a different notation.)*

Proposition 1.13. *Products of cuts is commutative.*

Lemma 1.14. *Let A, B be cuts. Then we have*

$$A(-B) = -(AB) = (-A)B.$$

Proposition 1.15. *Product of cuts is associative.*

Proposition 1.16. 1^* *is the multiplicative identity for cuts..*

Proposition 1.17 (Reciprocation of cuts)**.** *Let $A > 0^*$ be a cut. Then the set*

$$A^{-1} := \begin{cases} 0^* \cup \{0\} \cup (F \setminus A)^{-1}, & F \setminus A \text{ has no minimum} \\ 0^* \cup \{0\} \cup (F \setminus (A \cup \{L\})), & L \text{ is the minimum of } F \setminus A \end{cases}$$

is a cut, and it follows that

$$A^{-1} = 0^* \cup \{0\} \cup \{x > 0 : \text{for some } y \in F \setminus A, \text{we have } y < x^{-1}\}.$$

Using this, we define, for any $A \neq 0^*$,

$$A^{-1} := \begin{cases} A^{-1}, & A > 0^* \\ -((-A)^{-1}), & A < 0^* \end{cases}$$

which is again a cut.

Remark. *We again have abused notations here, two times.*

We can't extend the above definition to 0^* since that would yield $(0^*)^{-1} = F$, which isn't a cut.

Define exponentiation!

Lemma 1.18. *Let $x \in F$ such that $x > 1$. Then x^n can be made arbitrarily large for $n \in \mathbb{N}$.*

Proposition 1.19. *Reciprocation of a nonzero cut is its multiplicative inverse.*

Remark. *Proposition 1.19 is the second time that the Archimedean-ness of F is used (via Lemma 1.18).*

Theorem 1.20. *Nonzero cuts form an abelian multiplicative group.*

Theorem 1.21. *Cut multiplication distributes over cut addition.*

Remark. *Just showing for the all-positive does all the work!*

Theorem 1.22. *Order preserves multiplication, i.e.,*

$$A \leq B \text{ and } C \geq 0^* \implies AC \leq BC.$$

Remark. *We'll use Halmos' terminology for partially ordered sets.*

Theorem 1.23 (Least upper bound property)**.** *Let S be a nonempty set of cuts that is bounded above. Then*

$$\bigcup S$$

is the least upper bound of S.

Remark. *We'll use the usual notations for addition and multiplication operations on any set, and also for the inverses and identities therein.*

Definition 1.24 ((Complete) (ordered) fields)**.** Consider a set K with addition and multiplication. Then K is called a field iff each of the following hold:

(i) K is an abelian group under addition.

(ii) $K \setminus \{0\}$ is an abelian group under multiplication.

(iii) Multiplication distributes over addition.

We call K an ordered field iff it is a field along with a total order $\leq$ such that

(i) $a \leq b \implies a + c \leq b + c$, and

(ii) $a \leq b$ and $c \geq 0 \implies ac \leq bc$.

K is called a complete ordered field iff it is an ordered field with each nonempty bounded-above set in it having a least upper bound (*i.e.*, it is order-complete).

Show independence of axioms!

Theorem 1.25. *The set of cuts is a complete ordered field.*

Show the following!

Theorem 1.26. *(Embedding of) F is a subfield of the field of cuts.*

Theorem 1.27. *Any complete ordered field is isomorphic to the above field of cuts.*

Remark. *Hence, from now on, we'll fix a complete ordered field, $\mathbb{R}$, calling its elements, reals. We'll denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, the embeddings of appropriate sets, and call their elements as naturals, integers, rationals.*

2 Properties of $\mathbb{R}$

August 29, 2022

Remark. *Note that $\mathbb{R}$ is a field, and hence the usual algebraic results like $x0 = 0$, $(-1)x = -x$, et cetera will hold.*

Theorem 2.1. *Nonempty bounded-below sets have greatest lower bounds.*

Theorem 2.2. *$\mathbb{R}$ is Archimedean.*

Definition 2.3 (Monotone functions)**.** A function $f\colon X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$, is called weakly (respectively strictly) monotonically increasing iff for any $x, y \in \mathbb{X}$, we have $f(x) < f(y)$ (respectively $f(x) \leq f(y)$) whenever $x < y$.

We have similar definition for monotonically decreasing functions.

Remark. *We'll mean weak monotonicity when not specified.*

Theorem 2.4 (Floor and ceiling)**.** *Let $x \in \mathbb{R}$. Then there exist unique integers $\lfloor x \rfloor$ and $\lceil x \rceil$ such that*

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \text{ and } \lceil x \rceil - 1 < x \leq \lceil x \rceil.$$

Further, the following hold:

(i) *The functions $x \mapsto \lfloor x \rfloor$ and $x \mapsto \lceil x \rceil$ are monotonically increasing.*

(ii) $x \in \mathbb{Z} \implies \lfloor x \rfloor = x = \lceil x \rceil$.

(iii) $x \notin \mathbb{Z} \implies \lfloor x \rfloor < x < \lceil x \rceil$.

Definition 2.5 (Integer powers)**.** Since non-zero reals form a multiplicative group, we'll define their integer powers in the usual way.

We'll also define 0^n for $n \geq 0$: it being 1 for $n = 0$, and 0 otherwise.

Remark. *The properties of powers proved in Atul's Algebra will hold.*

Since the reals also form an additive group, we can also define nx's for integer n's (using the additive notation).

Lemma 2.6 ($\sqrt{2}$ is not rational)**.** *$x^2 \neq 2$ for any $x \in \mathbb{Q}$.*

Remark. *We'll use the usual $\sup A$ and $\inf$ notations.*

Theorem 2.7 ($\sqrt{2}$ is real)**.** *Let $A := \{x \in \mathbb{R} : x^2 \leq 2\}$, which is nonempty and bounded above. Let $\alpha := \sup A$. Then $\alpha \geq 0$ and*

$$\alpha^2 = 2.$$

Theorem 2.8. *Between any two reals is a rational as well as an irrational.*

Remark. *We'll use $\mathbb{R}^+$ as well as $\mathbb{R}^-$.*

Result 2.9. Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded above. Then $A + B$ and $-A$ are nonempty and bounded above, and

$$\sup(A + B) = \sup A + \sup B, \text{ and}$$
$$\sup(-A) = -\inf A.$$

If $A, B \subseteq \mathbb{R}^+ \cup \{0\}$, then AB is nonempty and bounded above, and

$$\sup(AB) = (\sup A)(\sup B).$$

If A is bounded below by some positive real, then A^{-1} is nonempty and bounded above, and

$$\sup(A^{-1}) = (\inf A)^{-1}.$$

3 Base representation for $\mathbb{R}$

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Proposition 3.1. *Let $r > 1$ be a real and $0 \leq d_{-1}, d_{-2}, \ldots < \lceil r \rceil$ be naturals. For $n \geq 0$, define*

$$D_n := \sum_{i=1}^{n} d_{-i} r^{-i}.$$

Then

$$0 \leq D_n < \frac{\lceil r \rceil - 1}{r - 1}$$

and we define

$$0.d_{-1}d_{-2}\ldots := \sup_{n \in \mathbb{N}} D_n.$$

It follows that

$$0.d_{-1}d_{-2}\ldots \in \left[\frac{\lceil r \rceil - 1}{r - 1}\right].$$

Remark. *Strictly speaking, we should also incorporate r in the notation.*

If the sequence $d_{-1}, d_{-2}, \ldots$ terminates (i.e., becomes zero after some point), then we can truncate the representation too.

Lemma 3.2. *Let $x, \varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 1$. Let (a_i) be a real sequence such that*

$$a_i < x \leq a_i + \varepsilon^i.$$

Then

$$\sup_i a_i = x.$$

Theorem 3.3 (Representation of $(0, 1]$)**.** *Let $r > 1$ be a real and $x \in (0, 1]$. Then there exist unique naturals $0 \leq d_{-1}, d_{-2}, \ldots < \lceil r \rceil$ such that for each $n \geq 0$, we have*

$$D_n < x \leq D_n + r^{-n}.$$

Further, we have that

$$x = 0.d_{-1}d_{-2}\ldots$$

and that the above is a non-terminating expansion.

Conversely, if $0 \leq e_{-1}, e_{-2}, \ldots < \lceil r \rceil$ are non-terminating, and if $r \in \mathbb{N}$, then

$$E_n < 0.e_{-1}e_{-2}\ldots \leq E_n + r^{-n}.$$

We further have that

$$0.e_{-1}e_{-2}\ldots \in (0, 1].$$

Remark. *For non-natural r, the converse breaks: For base ϕ, we have that $11 = 100$ and hence $0.1111\ldots = 0.1010\ldots$, which are both non-terminating.*

Definition 3.4. Let $r > 1$ be a real. Let $k \geq 0$ and $0 \leq n_0, \ldots, n_k < \lceil r \rceil$ be naturals. Then we define

$$n_k \ldots n_0 := \sum_{i=0}^{k} n_i r^i.$$

Remark. *Again, we should've incorporated r into the notation.*

Theorem 3.5 (Representing the integral parts for $[1, \infty)$)**.** *Let $r > 1$ and $x \geq 1$ be reals. Then there exist naturals $k \geq 0$ and $0 \leq n_0, \ldots, n_k < \lceil r \rceil$ such that $n_k \neq 0$ and*

$$0 \leq x - n_k \ldots n_0 < 1$$

and if $r \in \mathbb{N}$, then the above naturals are unique.

Remark. *For non-integer r, the uniqueness might break: For ϕ, we have $11 = 100$.*

Definition 3.6 (Representing $(0, \infty)$)**.** Let $r > 1$ be a real. Let $k \geq 0$ and $0 \leq n_0, \ldots, n_k, d_{-1}, d_{-2}, \ldots < \lceil r \rceil$ be naturals. Then we define

$$n_k \ldots n_0.d_{-1}d_{-2}\ldots := n_k \ldots n_0 + 0.d_{-1}d_{-2}\ldots.$$

Remark. *There was a possible notational collision for $0.d_{-1}d_{-2}\ldots$, but it does not happen since the above definition is a continuation of the previous one.*

Theorem 3.7. $\mathbb{R}$ *is uncountable.*

4 Absolute value—the norm on $\mathbb{R}$

August 30, 2022

Definition 4.1 (Absolute value)**.** For $x \in \mathbb{R}$, we define

$$|x| := \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

Proposition 4.2 (Properties of absolute values)**.** *Let $x, y \in \mathbb{R}$. Then the following hold:*

(i) $|x| \geq 0$.

(ii) $x = 0 \iff |x| = 0$.

(iii) $\big||x| - |y|\big| \leq |x + y| \leq |x| + |y|$.

(iv) $|-x| = |x|$.

(v) $|xy| = |x||y|$.

(vi) *If* $x \neq 0$, *then* $|x^n| = |x|^n$ *for* $n \in \mathbb{Z}$.

(vii) $|x| < y \iff -y < x < y$.

Chapter II

Sequences and series of reals

1 Sequences

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Definition 1.1 (Sequences)**.** Any function from an interval of $\mathbb{Z}$ to $\mathbb{R}$ is be called a real sequence. Depending on the finiteness of the domain interval, we'll call the sequence *finite* or *infinite*.

Notation. *That "a is a sequence" will be conveyed via these phrases as well:*

(i) "(a_i) (or $(a_i)_i$) is a sequence."

(ii) "$(a_i)_{i=m}^n$ is a sequence" where $m, n \in \mathbb{Z} \cup \{-\infty, +\infty\}$, when we want to mention the domain of a as well.

Definition 1.2 (Cauchy sequences)**.** A sequence[1] $(a_i)_{i=m}^\infty$ is said to be Cauchy iff for every[2] $\varepsilon > 0$, there exists an[3] $N \geq m$ such that for all[4] $i, j \geq N$, we have

$$|a_i - a_j| < \varepsilon.$$

Lemma 1.3 (Cauchy-ness blind to initial segments)**.** *Let $(a_i)_{i=m}^\infty$ be a sequence and $n \geq m$. Then the following are equivalent:*

(i) $(a_i)_{i=m}^\infty$ is Cauchy.

[1]Unless stated otherwise, such m's will be integers.

[2]Unless stated otherwise, ε will always represent reals.

[3]Obviously, N is an integer. We'll not make such remarks again.

[4]Here, i, j are integers.

(ii) $(a_{i+N})_{i=m}^{\infty}$ *is Cauchy.*[5]

(iii) $(a_i)_{i=n}^{\infty}$ *is Cauchy.*[6]

Definition 1.4 (Bounded sequences)**.** A (possibly finite) sequence (a_i) is said to be *bounded above* (respectively *bounded below*) iff there exists an[7] M such that each $a_i \leq M$ (respectively $a_i \geq M$).

Lemma 1.5. *Bounded-ness of sequences obeys an analogue of Lemma 1.3.*

Lemma 1.6 (Characterizing bounded-ness)**.** *A sequence* (a_i) *is bounded* $\iff$ $(|a_i|)$ *is bounded.*[8]

Definition 1.7 (Convergent sequences)**.** A sequence $(a_i)_{i=m}^{\infty}$ is said to converge to an $L \in \mathbb{R}$, denoted[9]

$$a_i \to L \text{ or } a_i \xrightarrow{i} L,$$

iff for each $\varepsilon > 0$, there exists an $N \geq m$ such that for all $i \geq N$, we have

$$|a_i - L| < \varepsilon.$$

Lemma 1.8. *Analogue of Lemma 1.3 holds for convergence of sequences to reals as well.*

Proposition 1.9. *For real sequences, we have*

$$\textit{convergent} \implies \textit{Cauchy} \implies \textit{bounded.}$$

Remark. *Theorem 2.9 is the converse of the first implication.*

Example 1.10. The sequence $((-1)^n)_n$ is bounded but neither convergent nor Cauchy.

Proposition 1.11. *A sequence can converge to at most one point.*

Notation. *Hence, if existent, we'll denote this limit by* $\lim_i a_i$ *or by* $\lim_{i\to\infty} a_i$.

[5] We are abusing notation here: We are actually talking about a sequence $(b_i)_{i=m}^{\infty}$ given by $b_i := a_{i+N}$.

[6] Again an abuse: This represents the restriction of the sequence.

[7] It doesn't matter if this M is an integer or a real.

[8] Abuse of notation: The sequence "$(|a_i|)$" is the sequence whose i-th element is $|a_i|$.

[9] Note that the the information about m is contained within a.

Theorem 1.12 (Limit manipulations)**.** *Let $a_i \to L$ and $b_i \to M$ in $\mathbb{R}$. Then the following hold:*[10]

(i) $a_i + b_i \to L + M$.

(ii) $\alpha a_i \to \alpha L$ *for any* $\alpha \in \mathbb{R}$.

(iii) $|a_i| \to |L|$.

(iv) $a_i b_i \to LM$.

(v) Each $a_i \neq 0$ *and* $L \neq 0 \implies a_i^{-1} \to L^{-1}$.

(vi) $a_i \to 0 \iff |a_i| \to |L|$.

(vii) Each $a_i \leq b_i \implies L \leq M$.

Theorem 1.13 (Sandwich)**.** *Let $a_i, c_i \to L$ and (b_i) be sandwiched between them, i.e., each $a_i \leq b_i \leq c_i$. Then $c_i \to L$ as well.*

Theorem 1.14 (Geometric sequences)**.** *Let $r \in \mathbb{R}$. Then we have the following cases:*

(i) $|r| < 1 \implies r^n \to 0$.

(ii) $|r| > 1 \implies (r^n)$ *is unbounded.*

Theorem 1.15 (Ratio test)**.** *Let (a_i) be a sequence with each $a_i \neq 0$ such that*

$$\left|\frac{a_{i+1}}{a_i}\right| \to L$$

in $\mathbb{R}$. Then we have the following cases:

(i) $L < 1 \implies a_i \to 0$.

(ii) $L > 1 \implies (a_i)$ *is unbounded.*

Result 1.16 ($\left((n!)^{1/n}\right)$ diverges)**.** For any real $C > 0$, we have that

$$\frac{C^n}{n!} \to 0,$$

and hence, we have that the sequence $\left((n!)^{1/n}\right)$ is unbounded.

Theorem 1.17 (Monotone convergence)**.** *A monotonically increasing (respectively decreasing) sequence (a_i) that is bounded above (respectively bounded below) is convergent, with*

$$a_i \to \sup_i a_i \text{ (respectively } a_i \to \inf_i a_i\text{).}$$

[10] Notation abuse: "$a_i + b_i$" is a sequence whose i-th element is $a_i + b_i$, *etc.*

Result 1.18 (Monotonic functions create monotone sequences)**.** Let $S \subseteq \mathbb{R}$ and $f\colon S \to \mathbb{R}$ be monotonically increasing with $f(S) \subseteq S$. Let $c \in S$ and define

$$a_0 := c, \text{ and}$$
$$a_{n+1} := f(a_n) \text{ for } n \geq 0.$$

Then (a_i) is monotonically increasing (respectively decreasing) if $a_0 \leq a_1$ (respectively $a_0 \geq a_1$).

If f were monotonically decreasing, then $f \circ f$ would be monotonically increasing and we'd have gotten interlaced monotonic sequences.

2 Subsequences

November 10, 2022

Definition 2.1 (Subsequences)**.** Let $(a_i)_{i=m}^{\infty}$ be a sequence. Let $n \in \mathbb{Z}$ and $f\colon \{n, n+1, \ldots\} \to \{m, m+1, \ldots\}$ be a strictly increasing function. Then the sequence $(a_{f(i)})_{i=n}^{\infty}$ is called a subsequence of $(a_i)_{i=m}^{\infty}$.

Lemma 2.2. *Subsequences of a convergent sequence converge to the same limit.*

Result 2.3. Monotone sequences having a convergent subsequence are convergent.

Result 2.4. Let $(a_{f_1(k)})_k, \ldots, (a_{f_n(k)})_k$ be subsequences of a sequence (a_i) for $n \geq 1$ such that the ranges of f_i's cover the domain of a and that $a_{f_1(k)}, \ldots, a_{f_n(k)} \to L$. Then $a_i \to L$.

Definition 2.5 ($\limsup$ and $\liminf$)**.** Let $(a_i)_{i=m}^{\infty}$ be a sequence. If it is bounded above, then we define the sequence $(a_k^+)_{k=m}^{\infty}$ via

$$a_k^+ := \sup_{i \geq k} a_i.$$

If $(a_i)_{i=m}^{\infty}$ is bounded below, then we also define the sequence $(a_k^-)_{k=m}^{\infty}$ via

$$a_k^- := \inf_{i \geq k} a_i.$$

If $(a_k^+)_{i=m}^\infty$ is bounded below, we define

$$\limsup_i a_i := \inf_{k\geq m} a_k^+,$$

and if $(a_k^-)_{i=m}^\infty$ is bounded above, we also define

$$\liminf_i a_i := \sup_{k\geq m} a_k^-.$$

Lemma 2.6. *An analogue of Lemma 1.3 also holds for* $\liminf$ *and* $\limsup$*.*

Theorem 2.7 (Bolzano-Weierstraß)**.** *Any bounded sequence has a subsequences converging to its* $\limsup$ *and* $\liminf$*.*

Proposition 2.8. *Let (a_i) be a bounded sequence and $L \in \mathbb{R}$. Then*

$$\limsup_i a_i = L = \limsup_i a_i \iff a_i \to L.$$

Theorem 2.9 ($\mathbb{R}$ is Cauchy-complete)**.** *Cauchy sequences in $\mathbb{R}$ are convergent.*

Proposition 2.10. *Let (a_i) be a sequence and $L \in \mathbb{R}$. Then $a_i \to L \iff$ every subsequence of (a_i) has a subsequence converging to L.*

Proposition 2.11 (Nested interval lemma)**.** *Let $I_1 \supseteq I_2 \supseteq \cdots$ be a nested sequence of nonempty closed intervals in $\mathbb{R}$. Then*

$$\bigcap_{n=1}^\infty I_n \neq \emptyset.$$

Result 2.12 (Convergence of contractive sequences)**.** Let $(a_i)_{i=0}^\infty$ be a sequence and $0 < \alpha < 1$ such that

$$|a_{i+1} - a_i| \leq \alpha |a_i - a_{i-1}|.$$

Then (a_i) is Cauchy with

$$\left|a_i - \lim_i a_i\right| \leq \frac{\alpha^i}{1-\alpha} |a_1 - a_0|$$

Remark. *Cauchy sequences needn't be contractive as $(1/n^2)$ shows. (See Theorem 3.6.)*

3 Series

August 31, 2022

Definition 3.1 (Convergence of series)**.** Let $(a_i)_{i=m}^{\infty}$ be sequence. Then we say that the associated series converges, written "$\sum_{i=m}^{\infty} a_i$ converges", iff the sequence $(s_k)_{k=m}^{\infty}$ of the partial sums, defined by

$$s_k := \sum_{i=m}^{k} a_i, \quad \text{for } k \geq m$$

converges, and in this case, we define

$$\sum_{i=m}^{\infty} a_i := \lim_k s_k.$$

If $(s_k)_{k=m}^{\infty}$ diverges, written "$\sum_{i=m}^{\infty} a_i$ diverges", we say that the associated series diverges.

Remark. *The initial integer is important here: Although the initial segments don't affect the limits of convergent* sequences, *they do affect the sum of the convergent series!*

Remark. *Since writing $\sum_{i=m}^{k} a_i \xrightarrow{k} S$ every time is cumbersome, we will just write that $\sum_{i=m}^{\infty} a_i = S$, omitting to mention that the partial sums are convergent.*

Lemma 3.2. *Let $(a_i)_{i=m}^{\infty}$ be sequence and $n \geq m$. Then we have that $\sum_{i=m}^{\infty} a_i$ converges $\iff$ $\sum_{i=n}^{\infty} a_i$ converges. And if they do, then*

$$\sum_{i=m}^{\infty} a_i = \sum_{i=m}^{n} a_i + \sum_{i=n+1}^{\infty} a_i.$$

Proposition 3.3. *Only sequences converging to zero can have convergent series.*

Theorem 3.4 (Manipulating series)**.** *Let $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ be convergent series. Then the following hold:*

(i) $\sum_{i=m}^{\infty}(a_i + b_i) = \sum_{i=m}^{\infty} a_i + \sum_{i=m}^{\infty} b_i$.

(ii) $\sum_{i=m}^{\infty}(\alpha a_i) = \alpha \sum_{i=m}^{\infty} a_i$ *for* $\alpha \in \mathbb{R}$.

(iii) *Each* $a_i \geq 0 \implies \sum_{i=m}^{\infty} a_i \geq 0$.
(iv) *Each* $a_i \leq b_i \implies \sum_{i=m}^{\infty} a_i \leq \sum_{i=m}^{\infty} b_i$.

Result 3.5 (Cesàro sums)**.** Let $(a_i)_{i=1}^{\infty}$ be a convergent sequence. Then

$$\frac{1}{n}\sum_{i=1}^{n} a_i \to \lim_i a_i.$$

Remark. *The series $((-1)^n)_n$ gives a counterexample to the converse.*

Theorem 3.6. *Let $p > 0$ be real. Then the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent for $p > 1$ and unbounded for $p \leq 1$.

Result 3.7. A sequence converging to 0 has a subsequence whose series is convergent.

Result 3.8. Let $(a_i)_{i=1}^{\infty}$ be a sequence. Then the following hold:
(i) $\sum_{i=1}^{\infty} |a_{i+1} - a_i|$ is convergent $\implies$ (a_i) is Cauchy.
(ii) (a_i) is Cauchy $\implies$ there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that the series $\sum_{k=1}^{\infty} |a_{n_{k+1}} - a_{n_k}|$ is convergent.

Remark. *Note that $(1/n)$ shows that $|a_{i+1} - a_i| \to 0$ is not sufficient for (a_i) to be Cauchy.*

Proposition 3.9. *Let $(a_i)_{i=1}^{\infty}$ and $(b_i)_{i=1}^{\infty}$ be sequences of positive reals such that $a_i/b_i \to L$ for an $L > 0$. Then $\sum_{i=1}^{\infty}$ converges $\iff$ $\sum_{i=1}^{\infty}$ converges.*

Chapter III

Continuity

Remark. *In this chapter, S, T, U will be some fixed subsets of $\mathbb{R}$.*

1 Topological aspects

November 10, 2022

Definition 1.1. For $\varepsilon > 0$ and $x \in \mathbb{R}$, we define

$$B_\varepsilon(x) := (x - \varepsilon, x + \varepsilon).$$

Definition 1.2 (Cluster points)**.** A point $c \in \mathbb{R}$ is called a cluster point of S iff for every $\varepsilon > 0$, we have that

$$B_\varepsilon(c) \cap S \setminus \{c\} \neq \emptyset.$$

We also set

$$\ell(S) := \{x \in \mathbb{R} : x \text{ is a cluster point of } S\}$$

Theorem 1.3 (Connection with limits of sequences)**.** *Let $c \in \mathbb{R}$. Then the following are equivalent:*[1]

(i) $c \in \ell(S)$.

(ii) There exists a sequence $(x_i) \in S \setminus \{c\}$ such that $x_i \to c$.

Proposition 1.4 (Connection with subsets)**.** *Let $S \subseteq T$ and $c \in \mathbb{R}$. Then the following hold:*

[1] "(i) $\Rightarrow$ (ii)" used CC.

(i) $c \in \ell(S) \implies c \in \ell(T)$.

(ii) *The converse of the above holds if* $S \supseteq B_r(c) \cap T$ *for some* $r > 0$.

Proposition 1.5. *For* $c \in \mathbb{R}$, *we have*

$$\ell(S) = \ell\big((-\infty, c) \cap S\big) \cup \ell\big(S \cap (c, +\infty)\big).$$

Corollary 1.6 (Removing finitely many points doesn't affect cluster points)**.** *Let* $c \in \mathbb{R}$. *Then*

$$\ell(S) = \ell(S \setminus \{c\}).$$

Definition 1.7 (Closed subsets of $\mathbb{R}$)**.** S is said to be closed iff every Cauchy sequence in S converges to some point in itself.

Corollary 1.8. *For* $a, b \in \mathbb{R}$, *closed intervals are closed.*

Definition 1.9 (Closure sets)**.** We define

$$\overline{S} := \{x \in \mathbb{R} : B_\varepsilon(x) \cap S \neq \emptyset \text{ for all } \varepsilon > 0\}.$$

Proposition 1.10.

(i) S *is closed* $\iff \overline{S} = S$.

(ii) $\overline{\overline{S}} = \overline{S}$.

(iii) $\overline{S} \setminus \ell(S) \subseteq S$.

(iv) $\overline{S} = S \cup \ell(S)$.

Proposition 1.11 (Intervals are connected)**.** *Let* $I \subseteq \mathbb{R}$. *Then the following are equivalent:*

(i) I *is an interval.*

(ii) *For any* $a, b, c \in \mathbb{R}$, *we have that* $a < b < c$ *and* $a, c \in I \implies b \in I$.

2 Limits of functions

November 10, 2022

Definition 2.1 (Limiting values of functions)**.** Let $f\colon S \to T$ and $c, L \in \mathbb{R}$. Then we write

$$f(x) \to L \text{ as } x \to c$$

iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$f\big(B_\delta(c) \cap S \setminus \{c\}\big) \subseteq B_\varepsilon(L).$$

Proposition 2.2 (Uniqueness of limits). *Let $f\colon S \to T$ and $c \in \mathbb{R}$. Then the following hold:*

(i) c is a cluster point of $\mathbb{R}$ $\implies$ there exists at most one $L \in \mathbb{R}$ such that $f(x) \to L$ as $x \to c$.

(ii) c is not a cluster point of S $\implies$ $f(x) \to L$ as $x \to c$ for every $L \in \mathbb{R}$.

Remark. *In case of c being a cluster point, this allows to denote the unique L (if existent) by $\lim_{x\to c} f(x)$.*

Theorem 2.3 (Limit manipulations). *Let $f, g\colon S \to T$ and $c, L, M \in \mathbb{R}$ such that $f(x) \to L$ and $g(x) \to M$ as $x \to c$. Then, as $x \to c$, the following hold:*[2]

(i) $f(x) + g(x) \to L + M$.

(ii) $\alpha f(c) \to \alpha L$ for any $\alpha \in \mathbb{R}$.

(iii) $|f(x)| \to |L|$.

(iv) $f(x)\, g(x) \to LM$.

(v) $f(x)^{-1} \to L^{-1}$ if $f(x) \neq 0$ for any $x \in S$, and $L \neq 0$.

(vi) $f(x) \to 0 \iff |f(x)| \to 0$.

(vii) Each $f(x) \leq g(x)$ and c is a cluster point of S $\implies$ $L \leq M$.

Theorem 2.4 (Sandwich). *Let $f, g, h\colon S \to T$ and $c, L \in \mathbb{R}$ such that $f(x), h(x) \to L$ as $x \to c$, and $f(x) \leq g(x) \leq h(x)$ for each $x \in S$. Then*

$$g(x) \to L \text{ as } x \to c.$$

Remark. *By "$(x_i) \in A$", we'll mean that (x_i) is a sequence that takes values in the set A.*

Theorem 2.5 (Connection with limits of sequences). *Let $f\colon S \to T$ and $c, L \in \mathbb{R}$. Then the following are equivalent:*[3]

(i) $f(x) \to L$ as $x \to c$.

(ii) For each $(x_i) \in S \setminus \{c\}$, we have that $x_i \to c \implies f(x_i) \to L$.

Proposition 2.6 (Connection with restrictions). *Let $f\colon S \to T$, and $X \subseteq S$ and $f(X) \subseteq Y$. Define $g\colon X \to Y$ as $x \mapsto f(x)$. Then, for $c, L \in \mathbb{R}$, the following hold:*

(i) $f(x) \to L$ as $x \to c \implies g(x) \to L$ as $x \to c$.

(ii) The converse of above holds if $X \supseteq B_r(c) \cap S$ for some $r > 0$.

[2] Note that we are at least being *consistent* in our abuse of notation.

[3] "(ii) ⇒ (i)" used CC.

3 Continuous functions

Definition 3.1 (Continuity)**.** Let $f\colon S \to T$ and $c \in S$. Then f is said to be *continuous at* c iff $f(x) \to f(c)$ as $x \to c$.

f is said to be *continuous* iff f is continuous at each $c \in S$.

Theorem 3.2 (Continuity preserving operations)**.** *Let $f, g\colon S \to T$ be continuous at $c \in S$. Then the following functions $S \to \mathbb{R}$ are also continuous at c:*

(i) $f(x) + g(x)$.

(ii) $\alpha f(x)$.

(iii) $|f(x)|$.

(iv) $f(x)\, g(x)$.

(v) $f(x)^{-1}$ *if* $f(x) \neq 0$ *for all* $x \in S$.

Proposition 3.3. *Constant and identity functions, and hence polynomials, are continuous on $\mathbb{R} \to \mathbb{R}$.*

Theorem 3.4 (Connection with limits of sequences)**.** *Let $f\colon S \to T$ and $c \in S$. Then the following are equivalent:*

(i) f *is continuous at* c.

(ii) *For every* $(x_i) \in S$, *we have that* $x_i \to c \implies f(x_i) \to f(c)$.

Theorem 3.5 (Compositions)**.** *Let $f\colon S \to T$ and $g\colon T \to U$. Let $c \in S$ and $L \in T$ with $f(x) \to L$ as $x \to c$ and g be continuous at L. Then*

$$g(f(x)) \to g(L) \text{ as } x \to c.$$

Remark. *We can't weaken the continuity hypothesis of g to $g(x) \to M$ as $x \to L$ to conclude that $g(f(x)) \to M$ as $x \to c$.*

Corollary 3.6 (Composition of continuous functions)**.** *Let $f\colon S \to T$ and $g\colon T \to U$ with f being continuous at $c \in S$ and g being continuous at $f(c)$. Then $g \circ f$ is continuous at c.*

Proposition 3.7 (Connection with restrictions)**.** *Let $f\colon S \to T$, and $X \subseteq S$ and $f(X) \subseteq Y$. Define $g\colon X \to Y$ as $x \mapsto f(x)$. Then, for $c \in X$, the following hold:*

(i) f *is continuous at* $c \implies g$ *is continuous at* c.

(ii) *The converse of the above holds if* $X \supseteq B_r(c) \cap S$ *for some* $r > 0$.

Theorem 3.8 (Extending a continuous function at limiting values[4]). *Let $f\colon S \to T$ be continuous and $T \subseteq \ell(S)$ such that $\lim_{x\to c} f(x)$ exists for each $c \in T$. Then there exists a unique continuous function $\hat{f}\colon S \cup T \to \mathbb{R}$ which is an extension of f.*

Further, this $\hat{f}$ is given by

$$\hat{f}(c) = \begin{cases} f(c), & c \in S \\ \lim_{x\to c} f(x), & c \in T \end{cases}$$

Result 3.9. A continuous automorphism on $(\mathbb{R}, +)$ is of the form $x \mapsto \alpha x$ for some $\alpha \in \mathbb{R}$.

Theorem 3.10 (Pasting lemma[5]). *Let S, T be closed, and $f\colon S \to \mathbb{R}$ and $g\colon T \to \mathbb{R}$ be continuous with f, g agreeing on $S \cap T$. Then the pasted function $h\colon S \cup T \to \mathbb{R}$ given by*

$$h(x) = \begin{cases} f(x), & x \in S \\ g(x), & x \in T \end{cases}$$

is continuous.

Result 3.11. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous at $c \in \mathbb{R}$. Define $h\colon \mathbb{R} \to \mathbb{R}$ as

$$x \mapsto \begin{cases} f(x), & x \in \mathbb{Q} \\ g(x), & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

Then for any $L \in \mathbb{R}$, the following are equivalent:

(i) h is continuous at c with $h(c) = L$.

(ii) $f(c) = L = g(c)$.

4 Uniform and Lipschitz continuities

November 10, 2022

Definition 4.1 (Uniform continuity). A function $f\colon S \to T$ is called uniformly continuous iff for every $\varepsilon > 0$, there exists a δ such that for all $x, y \in S$, we have that

$$|x - y| < \delta \implies |f(x) - f(y)| \leq \varepsilon.$$

[4] This used CC.

[5] CC used.

Definition 4.2 (Lipschitz continuity)**.** A function $f\colon S \to T$ is called Lipschitz continuous iff there exists an $\alpha > 0$ such that for all $x, y \in S$, we have that

$$|f(x) - f(y)| \leq \alpha\, |x - y|.$$

Proposition 4.3. *For a function $f\colon S \to T$, we have*

Lipschitz continuous $\implies$ uniformly continuous $\implies$ continuous.

Proposition 4.4. *Uniformly (respectively Lipschitz) continuous functions are closed under addition and scalar multiplication.*

Theorem 4.5. *Uniform continuity preserves Cauchy-ness of sequences.*

Remark. *Cauchy-ness is not preserved for continuous functions in general: Consider $(1/n)_n$ under the function $x \mapsto 1/x$.*

Theorem 4.6. *A continuous function with a closed and bounded domain is uniformly continuous.*[6]

Remark. *For a function $f\colon S \to T$ and $c \in \ell(S)$, we'll write "$\lim_{x\to c} f(x)$ exists" to mean that there exists an $L \in \mathbb{R}$ such that $f(x) \to L$ as $x \to c$.*

Theorem 4.7 (Connection with limiting values)**.** *Let $f\colon S \to T$ be continuous. Then the following hold:*[7]

(i) f is uniformly continuous $\implies$ $\lim_{x\to c} f(x)$ exists for each $c \in \ell(S)$.

(ii) The converse of the above holds if S is bounded.

5 Three important theorems

November 10, 2022

[6] CC is used.
[7] CC used in both.

Theorem 5.1 (Contraction mapping). *Let $S \neq \emptyset$ be closed, $f\colon S \to S$ and $0 < \alpha < 1$ be such that*

$$|f(x) - f(y)| \leq \alpha |x - y|.$$

Then there exists a unique fixed point $c \in S$, i.e., $f(c) = c$.

Further, for $\alpha \in S$, if we define a sequence $(x_i) \in S$ as

$$x_0 := \alpha, \text{ and}$$
$$x_{i+1} := f(x_i) \text{ for } i \geq 0,$$

then $x_i \to c$ contractively.[8]

Remark. *"$f\colon S \to T$ achieves its maximum (respectively minimum) on S" will mean that $f(S)$ is bounded above (respectively bounded below) and that there exists an $x \in S$ such that $f(x) \leq f(x)$ (respectively $f(x) \geq f(x)$) for each $x \in S$.*

Theorem 5.2 (Extreme value[9]). *Let $S \neq \emptyset$ be closed and bounded and $f\colon S \to T$ be continuous. Then f achieves its maximum and minimum on S.*

Corollary 5.3. *There doesn't exist any continuous bijection from a closed bounded interval to an open or a half-open-half-closed interval.*

Lemma 5.4. *Let $a < b$ be reals and $f\colon [a, b] \to S$ be continuous with $f(a) < 0 < f(b)$. Then define sequences (a_i), (b_i) as follows:*

$$(a_0, b_0) := (a, b), \text{ and}$$
$$(a_{i+1}, b_{i+1}) := \begin{cases} \left(\frac{a_i+b_i}{2}, b_i\right), & f\left(\frac{a_i+b_i}{2}\right) < 0 \\ \left(a_i, \frac{a_i+b_i}{2}\right), & f\left(\frac{a_i+b_i}{2}\right) \geq 0 \end{cases} \text{ for } i \geq 0.$$

Then (a_i), (b_i) converge contractively to some same fixed point of f.

Remark. *By "b lies between a and c", we'll mean that either $a \leq b \leq c$ or $c \leq b \leq a$. We'll also use "strict" in the usual sense here.*

Theorem 5.5 (Intermediate value). *Let $a < b$ be reals and $f\colon [a, b] \to S$ be continuous. Let y_0 lie strictly between $f(a)$ and $f(b)$. Then there exists an $x_0 \in (a, b)$ such that*

$$f(x_0) = y_0.$$

[8] Meaning that the sequence (x_i) is contractive. (See Result 2.12.)
[9] CC used.

Corollary 5.6. *Continuous functions map intervals to intervals.*[10]

Lemma 5.7 (Characterizing monotonicity)**.** *Let $|S| \geq 3$ and $f\colon S \to T$ not be strictly monotonic. Then there exist $a < b < c \in S$ such that $f(b)$ does not lie strictly between $f(a)$ and $f(c)$.*

Corollary 5.8. *A continuous function on an interval (having more more than one*[11] *element) is injective $\iff$ it is strictly monotonic.*

6 Polynomials

November 11, 2022

Remark. *By a "polynomial", unless stated otherwise, we'll mean the function determined by the formal polynomial.*

Our polynomials will be over $\mathbb{R}$ unless stated otherwise.

Theorem 6.1 (Asymptotic behaviour depends on the degree)**.** *Let $a_0, \ldots, a_n \in \mathbb{R}$ for $n \geq 0$. Then there exists an $N \geq 1$ such that for each $x \in \mathbb{R}$, we have that*

$$|x| \geq N \implies |x^{n+1}| > |a_0| + \cdots + |a_n x^n|.$$

Theorem 6.2. *A polynomial of odd degree has at least one root in $\mathbb{R}$.*

Theorem 6.3. *A bounded polynomial is constant.*

Corollary 6.4 (Polynomial determines its coefficients)**.** *Let $a_0, \ldots, a_n \in \mathbb{R}$ for $n \geq 0$ and define $p\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto a_0 + \cdots + a_n x^n$. Suppose $p(x) = 0$ for each $x \in \mathbb{R}$. Then each $a_i = 0$.*

Result 6.5. Polynomials are Lipschitz continuous on $[0, 1]$.

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[10] Theorem 8.2 gives a converse for monotones.

[11] And hence infinitely many.

7 One-sided limits

November 11, 2022

Notation. *If $f\colon S \to T$, then for $X \subseteq S$, we'll write $f|_X$ to be the restriction of f on $X \to T$.*

Definition 7.1 (One-sided limits)**.** Let $f\colon S \to T$ and $c, L \in \mathbb{R}$. Then we write

(i) "$f(x) \to L$ as $x \to c^-$" iff $f|_{(-\infty,c)\cap S}(x) \to L$ as $x \to c$.

(ii) "$f(x) \to L$ as $x \to c^+$" iff $f|_{S\cap(c,+\infty)}(x) \to L$ as $x \to c$.

Further, if $c \in \ell\big((-\infty, c) \cap S\big)$, then we'll denote $\lim_{x\to c} f|_{(-\infty,c)\cap S}(x)$, if existent, by

$$\lim_{x\to c^-} f(x).$$

Similarly, we'll use

$$\lim_{x\to c^+} f(x)$$

for $\lim_{x\to c} f|_{S\cap(c,+\infty)}(x)$, if existent, for $c \in \ell\big(S \cap (c, +\infty)\big)$.

Corollary 7.2. *The analogue of Theorem 2.3 holds for one-sided limits (with appropriate modification in the last point there).*

Lemma 7.3 (Connection with usual limits)**.** *Let $f\colon S \to T$ and $c, L \in \mathbb{R}$. Then the following are equivalent:*

(i) $f(x) \to L$ as $x \to c$.

(ii) $f(x) \to L$ as $x \to c^-$, and as $x \to c^+$.

Corollary 7.4 (Connection with continuity)**.** *Let $f\colon S \to T$ and $c \in S$. Then the following are equivalent:*

(i) f is continuous at c.

(ii) $f(x) \to f(c)$ as $x \to c^-$, and as $x \to c^+$.

8 Monotone functions

November 11, 2022

Remark. *For an $A \subseteq \mathbb{R}$, when we say "$\sup A$ exists", we mean that $A \neq \emptyset$ and that A is bounded above.*

Theorem 8.1 (Bounded monotones have one-sided limits)**.** *Let $f\colon S \to T$ be monotonically increasing and $c, d \in \mathbb{R}$ such that*

$$L^- := \sup\{f(x) : x \in (-\infty, c) \cap S\}, \text{ and}$$
$$L^+ := \inf\{f(x) : x \in S \cap (d, +\infty)\}$$

exist. Then the following hold:

(i) $f(x) \to L^-$ *as* $x \to c^-$, *and* $f(x) \to L^+$ *as* $x \to d^+$.

(ii) $c \leq d \implies L^- \leq L^+$.

(iii) $c > d \implies L^- \geq L^+$.

Remark. *Similar proposition holds for monotonically decreasing f.*

Theorem 8.2. *Let I be an interval and $f\colon I \to S$ be monotone with $f(I)$ being an interval. Then f is continuous.*

Corollary 8.3 (A source for homeomorphisms)**.** *Let I, J be intervals and $f\colon I \to J$ be a strictly monotonic surjection. Then f and f^{-1} are continuous.*

Theorem 8.4. *A monotone function can have at most countably many discontinuities.*[12]

[12] AC used here.

Chapter IV

Differentiability

Remark. *In this chapter too, we'll take S, T, U to be some fixed subsets of $\mathbb{R}$. We'll also take I, J to be some fixed intervals of $\mathbb{R}$.*

1 Taking derivatives

November 11, 2022

Definition 1.1 (Derivative)**.** Let $f\colon S \to T$ and $c \in S$. Define the *quotient function* $\tilde{f}\colon S \setminus \{c\} \to \mathbb{R}$ as

$$\tilde{f}(x) := \frac{f(x) - f(c)}{x - c}.$$

If $c \in \ell(S \setminus \{c\})$ (or equivalently,[1] $c \in \ell(S)$), then, if existent, we set

$$f'(c) := \lim_{x \to c} \tilde{f}(x)$$

and say that f is *differentiable* at c with $f'(c)$ being its *derivative*.

For a subset $X \subseteq S \cap \ell(S)$, we say that f *is differentiable on* X iff f is differentiable at each $c \in X$.

We also say that f is *differentiable* iff f is differentiable on $S \cap \ell(S)$.

Lemma 1.2 (Differentiability and continuity of the quotient function)**.** *Let $f\colon S \to T$. Let $c \in S \cap \ell(S)$ and $L \in \mathbb{R}$. Define $\underset{\sim}{f}\colon S \to \mathbb{R}$ as*

$$\underset{\sim}{f}(x) := \begin{cases} \frac{f(x)-f(c)}{x-c}, & x \neq c \\ L, & x = c \end{cases}.$$

[1] See Corollary 1.6.

Then $\underset{\sim}{f}$ is continuous at $c \iff f$ is differentiable at c with $f'(c) = L$.

Theorem 1.3 (Differentiability $\implies$ continuity). *Let $f\colon S \to T$ be differentiable at $c \in S \cap \ell(S)$. Then f is continuous at c.*

Remark. *For a function $f\colon S \to T$ and $c \in S \cap \ell(S)$, by "$f'(c) = L$", we'll mean that $f'(c)$ exists and equals L.*

Theorem 1.4 (Manipulating derivatives). *Let $f, g\colon S \to T$ be differentiable at $c \in S \cap \ell(S)$. Then the following hold:*

(i) $\big(f(x) + g(x)\big)'(c) = f'(c) + g'(c)$.

(ii) $\big(\alpha f(x)\big)'(c) = \alpha f'(c)$.

(iii) $\big(f(x)\, g(x)\big)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$.

(iv) $\big(f(x)/g(x)\big)'(c) = \big(f'(c)\, g(c) - f(c)\, g'(c)\big)/g(c)^2$ *if $g(x) \neq 0$ for all $x \in S$.*

Theorem 1.5 (Chain rule). *Let $f\colon S \to T$ be differentiable at $c \in S \cap \ell(S)$ and $g\colon T \to U$ be differentiable at $f(c) \in T \cap \ell(T)$. Then*

$$\big(g(f(x))\big)'(c) = g'(f(c))\, f'(c).$$

Theorem 1.6 (Derivative of restrictions). *Let $f\colon S \to T$, and $X \subseteq S$ and $f(X) \subseteq Y$. Define $g\colon X \to Y$ as $x \mapsto f(c)$. Let $c \in X \cap \ell(X)$. Then the following hold:*

(i) f is differentiable at $c \implies g$ is differentiable at c with $g'(c) = f'(c)$.

(ii) The converse of the above holds if $X \supseteq B_r(c) \cap S$ for some $r > 0$.

Proposition 1.7. *Constant and identity functions are differentiable with their deruvatives being 0 and 1 respectively.*

Proposition 1.8 (Derivative of monomials). *Let $n \geq 1$ and $f\colon \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^n$. Then f is differentiable with the derivative given by*

$$f'(x) = nx^{n-1}$$

for $x \in \mathbb{R}$.

Result 1.9. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable at $c \in \mathbb{R}$. Define $h\colon \mathbb{R} \to \mathbb{R}$ as

$$x \mapsto \begin{cases} f(x), & x \in \mathbb{Q} \\ g(x), & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

Then for any $L \in \mathbb{R}$, the following are equivalent:

(i) h is differentiable at c with $h'(c) = L$.

(ii) $f'(c) = L = g'(c)$.

Proposition 1.10 (Lipschitz continuity and the derivative)**.** *Let $f: I \to S$ be differentiable. Then f is Lipschitz continuous iff f' is bounded.*[2]

Proposition 1.11 (A version of l'Hôpital)**.** *Let $f, g: S \to T$ be differentiable at $c \in S \cap \ell(S)$ with $f(c) = 0 = g(c)$ but with $g'(c) \neq 0$ and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$. Then*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

2 Mean value theorems

November 16, 2022

Definition 2.1 (Local extrema)**.** A point $c \in \mathbb{R}$ is called a point of local minimum (respectively local maximum) iff there exists an $\varepsilon > 0$ such that for each $x \in B_\varepsilon(c) \cap S$, we have

$$f(x) \geq f(c) \text{ (respectively } f(x) \leq f(c)\text{)}.$$

We call c a point of *local extremum* iff it is either a point of local minimum or of local maximum.

Theorem 2.2 (Slope test for local extrema)**.** *Let $f: S \to T$ and $c \in \ell\big((-\infty, c) \cap S\big) \cap S \cap \ell\big(S \cap (c, +\infty)\big)$ be point of local extremum. Let f be differentiable at c. Then*

$$f'(c) = 0.$$

Theorem 2.3 (Rolle's theorem)**.** *Let $a < b$ in $\mathbb{R}$, and $f: [a, b] \to S$ be continuous, and differentiable on (a, b) with $f(a) = f(b)$. Then there exists a $c \in (a, b)$ such that*

$$f'(c) = 0.$$

Theorem 2.4 (Cauchy's mean value)**.** *Let $a < b$ and $f, g: [a, b] \to S$ be continuous, and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that*

$$\big(f(b) - f(a)\big)g'(c) = f'(c)\big(g(b) - g(a)\big).$$

[2] Of course, by f', we mean the function $x \mapsto f'(x)$. See Definition 3.1.

Corollary 2.5 (Mean value)**.** *Let $a < b$ in $\mathbb{R}$, and $f\colon [a,b] \to S$ be continuous, and differentiable on (a,b). Then there exists a $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 2.6. *Let I be an interval and $f\colon I \to S$ be differentiable such that $f'(x) = 0$ for each $x \in I$. Then f is constant over I.*

Corollary 2.7 (Connection with monotonicity)**.** *Let I be an interval and $f\colon I \to S$ be differentiable. Then the following are equivalent:*

(i) $f'(x) \geq 0$ (respectively $f'(x) \leq 0$) for each $x \in I$.

(ii) f is increasing (respectively decreasing).

Further, strictness is preserved in "(i) $\Rightarrow$ (ii)" direction.

Corollary 2.8. *Let I be an interval and $f\colon I \to S$ be continuous, and differentiable on $I \setminus \{c\}$ where $c \in I$. Let $f'(x) \to L$ as[3] $x \to c$. Then f is differentiable at c with $f'(c) = L$.*

Theorem 2.9 (Intermediate value property for derivatives)**.** *Let $a < b$ in $\mathbb{R}$ and $f\colon [a,b] \to S$ be differentiable. Let y_0 lie strictly between $f'(a)$ and $f'(b)$. Then there exists a $c \in (a,b)$ such that*

$$f'(c) = y_0.$$

Example 2.10 (Discontinuous derivative)**.** Define $f\colon \mathbb{R} \to \mathbb{R}$ as[4]

$$x \mapsto \begin{cases} (x\sin(1/x))^2, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then f is differentiable and Lipschitz continuous on $\mathbb{R}$, but the f' is discontinuous[5] at 0.

3 Taylor's theorem

November 16, 2022

[3] Of course, the function here is $x \mapsto f'(x)$ on $I \setminus \{c\} \to \mathbb{R}$.

[4] We haven't defined sin yet, but we just need the periodicity and differentiability of this function.

[5] Of course, we are talking about the function $x \mapsto f'(x)$. See Definition 3.1.

Definition 3.1 (n-th derivatives)**.** Let $f: S \to T$. Then we inductively define the *n-th derivative functions* $f^{(n)}$'s with codomain $\mathbb{R}$ as follows:

(i) $\operatorname{dom} f^{(0)} := S$ with $x \mapsto f(x)$.

(ii) $\operatorname{dom} f^{(n+1)} := \{c \in \operatorname{dom} f^{(n)} \cap \ell(\operatorname{dom} f^{(n)}) : f^{(n)} \text{ is differentiable at } c\}$ with $x \mapsto (f^{(n)})'(x)$.

We say that f is *n times differentiable at a* $c \in \mathbb{R}$ iff $c \in \operatorname{dom} f^{(n)}$.

We say that f is *n times differentiable on an* $X \subseteq \mathbb{R}$ iff $X \subseteq \operatorname{dom} f^{(n)}$.

We say that f is *n times differentiable* iff

$$\operatorname{dom} f^{(i+1)} = \operatorname{dom} f^{(i)} \cap \ell(\operatorname{dom} f^{(i)}) \text{ for all } i < n.$$

Proposition 3.2. *Let $f: S \to T$. Then for any $m, n \geq 0$, we have*

$$\left(f^{(m)}\right)^{(n)} = f^{(m+n)}.$$

Theorem 3.3 (Linearity of n-th derivatives)**.** *Let I be an interval and $f, g: I \to S$ be n times differentiable for an $n \geq 0$. Let $\alpha \in \mathbb{R}$. Then the following hold:*[6]

(i) $f + g$ *and* αf *are n times differentiable.*

(ii) $\operatorname{dom} f^{(n)}$, $\operatorname{dom} g^{(n)}$, $\operatorname{dom}(f+g)^{(n)}$, $\operatorname{dom}(\alpha f)^{(n)}$ *are all I.*

(iii) $(f+g)^{(n)} = f^{(n)} + g^{(n)}$.

(iv) $(\alpha f)^{(n)} = \alpha f^{(n)}$.

Proposition 3.4 (n-th derivative of monomials)**.** *Let $n, i \geq 0$ and $f: \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^n$. Then f is i times differentiable with $\operatorname{dom} f^{(i)} = \mathbb{R}$ and is given by*

$$f^{(i)}(x) = \begin{cases} \frac{n!}{(n-i)!}\, x^{n-i}, & i \leq n \\ 0, & i > n \end{cases}.$$

Theorem 3.5 (Taylor's polynomials agree nicely)**.** *Let $f: S \to T$ be n times differentiable at*[7] *$x_0 \in \mathbb{R}$ for an $n \geq 0$. Define the corresponding Taylor polynomial $p: \mathbb{R} \to \mathbb{R}$ as*

$$x \mapsto \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i.$$

Then for each $0 \leq i \leq n$, we have

$$p^{(i)}(x_0) = f^{(i)}(x_0).$$

[6] Here we have broken our consistency of using the imprecise notation by using the precise one.

[7] This means that $x_0 \in \operatorname{dom} f^{(n)}$. See Definition 3.1.

Remark. *We'll say "$f\colon S \to T$ is n-times continuously differentiable" iff f is n times differentiable with $f^{(i)}$ being continuous for each $i \leq n$.*

Theorem 3.6 (Taylor[8])**.** *Let $a < b$ in $\mathbb{R}$ and $f\colon [a,b] \to S$ be n times continuously differentiable, and $n+1$ times differentiable on (a,b). Then there exists a $c \in (a,b)$ such that*

$$f(b) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(b-a)^i + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Remark. *We'll use f'' for $f^{(2)}$, etc.*

Corollary 3.7 (Second derivative test)**.** *Let $a < b$ in $\mathbb{R}$ and $f\colon (a,b) \to S$ be twice continuously differentiable. Let $c \in (a,b)$ such that $f(c) = 0$ and $f''(c) > 0$ (respectively $f''(c) < 0$). Then c is a point of strict local minimum (respectively maximum).*

Corollary 3.8. *The n-th degree Taylor polynomial of a polynomial function f is equal to f.*

Corollary 3.9 (Taylor polynomials as good approximations)**.** *Let $a < b$ in $\mathbb{R}$ and $f\colon [a,b] \to S$ be n times continuously differentiable for $n \geq 0$. Then there exists a $\lambda \geq 0$ such that*

$$\left| \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i - f(x) \right| \leq \lambda |x-a|^n$$

for all $x \in [a,b]$.

Example 3.10 (Solving an ODE)**.** Let $a, b, c \in \mathbb{R}$. Then there exists a unique differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = a$, $f'(0) = b$, and $f''(x) = c$ for each $x \in \mathbb{R}$. It is given by

$$x \mapsto a + bx + \frac{c}{2}x^2.$$

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[8] Corollary 2.5 falls out if we set $n = 0$.

4 Inverse function theorem

November 16, 2022

Theorem 4.1. *Let I, J be intervals and $f\colon I \to J$ be strictly monotonic and surjective.*[9] *Let f be differentiable at $c \in I$ with $f'(c) \neq 0$. Then f^{-1} is differentiable at $f(c)$ with*

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}.$$

Further, if f is continuously differentiable with $f'(x) \neq 0$ for all $x \in I$, then f^{-1} is too.

Remark. *By an endpoint of an interval, we'll mean its* sup *or* inf*, if existent. Thus, the only endpoint of $(-\infty, 1)$ is 1.*

Theorem 4.2 (Inverse function)**.** *Let I be an interval and $f\colon I \to S$ be continuously differentiable. Let $c \in I$ such that c is not an endpoint of I and f is differentiable at c with $f'(c) \neq 0$. Then there exists an open interval J such that $c \in J \subseteq I$, and the function $\check{f}\colon J \to f(J)$ defined by $x \mapsto f(x)$ is invertible with the inverse being continuously differentiable, given by*

$$(\check{f}^{-1})'(y) = \frac{1}{f'(\check{f}^{-1}(y))}.$$

Example 4.3 (Necessity of "continuously differentiable")**.** The function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} x + 2x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is (discontinuously) differentiable at 0 with $f'(0) \neq 0$ and still we can't invert it in any neighbourhood of 0.

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[9] This already means that f and f^{-1} are continuous. See Theorem 8.2.

5 One-sided derivatives

November 17, 2022

Definition 5.1 (One-sided derivatives)**.** Let $f\colon S \to T$. Then the one-sided limits, if existent, of the quotient function at a $c \in S \cap \ell(S)$, as defined in Definition 1.1, are called the one-sided derivatives of f.

If existent, these will be denoted by $f'_-(c)$ and $f'_+(d)$ for $c \in \ell\big((-\infty, c) \cap S\big) \cap S$ and $d \in S \cap \ell\big(S \cap (c, +\infty)\big)$.

Theorem 5.2 (Connection with continuity and differentiability)**.** *Let $f\colon S \to T$ and $c \in \ell\big((-\infty, c) \cap S\big) \cap S \cap \ell\big(S \cap (c, +\infty)\big)$ with $f'_-(c)$, $f'_+(c)$ existent. Then the following hold:*

(i) f is continuous at c.

(ii) $f'_-(c) = f'_+(c) \implies f$ is differentiable at c.

6 Convex functions

November 17, 2022

Definition 6.1 (Convex functions)**.** Then a function $f\colon I \to S$ is called convex iff for all $x, y \in I$ and for each $0 \le t \le 1$, we have that

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y).$$

Lemma 6.2 (Slopes of convex functions monotonically increase)**.** *Let $f\colon I \to S$ be convex. Then for any $a < b < c$ in I, we have that*

$$\frac{f(b) - f(a)}{b - a} \le \frac{f(c) - f(a)}{c - a} \le \frac{f(c) - f(b)}{c - b}.$$

Theorem 6.3 (Convex functions on intervals have one-sided derivatives)**.** *Let $f\colon I \to R$ be convex and $x < y$ in I not be the endpoints of I. Then the following hold:*

(i) f has both, left- and right-hand derivatives existent at x, y. In particular, f is continuous at x, y.[10]

(ii) $f'_-(x) \le f'_+(x)$.

(iii) $f'_+(x) \le f'_-(y)$.

[10] Note that f *can* be discontinuous at the endpoints.

Theorem 6.4 (Characterizing convexity)**.** *Let $f\colon I \to S$ be differentiable. Then the following are equivalent:*

(i) f *is convex.*

(ii) $f(y) - f(x) \geq f'(x)(y - x)$ *for each* $x, y \in I$*.*

(iii) f' *is monotonically increasing.*

Further, if f *is twice differentiable, then we further have that* f *is convex* $\iff$ $f''(x) \geq 0$ *for each* $x \in I$*.*

Chapter V

The Riemann integral

Remark. *In this chapter, we'll again let S, T, U be some generic subsets of $\mathbb{R}$. We will also take a, b to be some general reals.*

1 Darboux sums

November 19, 2022

Definition 1.1 (Partitions of closed bounded intervals)**.** Let $a \leq b$. Then we call a finite sequence $(x_i)_{i=0}^n$, for $n \geq 0$, a *partition of* $[a, b]$ iff it is strictly monotonic with

$$x_0 = a, \text{ and}$$
$$x_n = b.$$

We will also identify the set $\{x_0, \ldots, x_n\}$ with the sequence (x_i) for partitions.[1]

Definition 1.2 (Lower and upper sums)**.** Let $a \leq b$ and $f\colon [a, b] \to S$ be bounded by m and M so that

$$m \leq f(x) \leq M$$

for each $x \in [a, b]$. Let $(x_i)_{i=0}^n$ be a partition of $[a, b]$. Define, for $1 \leq i \leq n$,

$$\begin{aligned} m_i &:= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i &:= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \text{ and} \\ \Delta x_i &:= x_{i-1} - x_i. \end{aligned}$$

[1] This is not very perverse since partitions are strictly increasing sequences.

Then we define the *lower* and *upper sums* of f with respect to the partition (x_i) to respectively be

$$L(f,(x_i)) := \sum_{i=1}^{n} m_i\,\Delta x_i, \text{ and}$$
$$U(f,(x_i)) := \sum_{i=1}^{n} M_i\,\Delta x_i.$$

Remark. *When we say that "$f\colon S \to T$ is bounded on $X \subseteq S$ by m and M", we mean that $m \le f(x) \le M$ for each $x \in X$.*

Proposition 1.3 (Lower and upper sums of a bounded function are bounded)**.** *Let $f\colon S \to T$ and $[a,b] \subseteq S$ for $a \le b$ such that f is bounded on $[a,b]$ by m and M. Let P be a partition of $[a,b]$. Then we have*

$$m(b-a) \le L\big(f|_{[a,b]}, P\big) \le U\big(f|_{[a,b]}, P\big) \le M(b-a).$$

Remark. *This allows to talk of the* lower *and* upper integrals *of f on $[a,b]$:*

$$\overline{\int_a^b} f := \inf\big\{U(f,P) : P \text{ is a partition of } [a,b]\big\}, \text{ and}$$
$$\underline{\int_a^b} f := \sup\big\{L(f,P) : P \text{ is a partition of } [a,b]\big\}.$$

Lemma 1.4. *Let $l\colon \mathbb{N} \to \mathbb{N}$ be strictly increasing with $l_0 = 0$. Let $x_1, \dots, x_n \in \mathbb{R}$ for $n \ge 0$. Let $k \ge 0$ and $l_k \le n$. Then*

$$\sum_{i=1}^{k}\Bigg(\sum_{j=l_{i-1}+1}^{l_i} x_j\Bigg) = \sum_{i=l_0+1}^{l_k} x_i.$$

Proposition 1.5 (Refinements' effect on lower and upper sums)**.** *Let $a \le b$ and $f\colon [a,b] \to S$ be bounded. Let $P \subseteq Q$ be partitions of $[a,b]$. Then we have*

$$L(f,P) \subseteq L(f,Q), \text{ and}$$
$$U(f,P) \supseteq U(f,Q).$$

Corollary 1.6. *For a bounded function on a closed bounded interval, any lower sum is less than or equal to any upper sum.*

Corollary 1.7. *Let $a \le b$ and $f\colon [a,b] \to S$ be bounded by m and M. Then we have*

$$m(b-a) \;\le\; \underline{\int_a^b} f \;\le\; \overline{\int_a^b} f \;\le\; M(b-a).$$

Proposition 1.8. *Changing the function at finitely many points has no effect on lower and upper integrals.*

Definition 1.9 (Riemann integrability)**.** Let $f\colon S \to T$ be bounded on $[a,b] \subseteq S$ where $a \le b$. Then we say that f is *Riemann integrable on* $[a,b]$ iff

$$\underline{\int_a^b} f = \overline{\int_a^b} f,$$

in which case, we denote the above by

$$\int_a^b f,$$

and also define[2]

$$\int_b^a f := -\int_a^b f.$$

Further, if $S = [a,b]$, then f will be called *Riemann integrable* iff f is Riemann integrable on $[a,b]$.

Example 1.10 (A non-Riemann-integrable function)**.** For the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) := \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable:

$$\underline{\int_0^1} f = 0 < 1 = \overline{\int_0^1} f.$$

Theorem 1.11 (Riemann condition)**.** *Let $a \le b$ and $f\colon [a,b] \to S$. Then the following are equivalent:*

[2] There's an apparent overloading for the case $a = b$. But it's harmless since $-0 = 0$.

(i) f is Riemann integrable on $[a, b]$.
(ii) For each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Remark. *Unless mentioned otherwise, for a partition (x_i), we'll take Δx_i's to be as defined as in Definition 1.2.*

2 Connection with Riemann sums

November 22, 2022

Definition 2.1 (Riemann sums)**.** Let $a \leq b$ and $f\colon [a, b] \to S$. Let $(x_i)_{i=1}^n$ be a partition of $[a, b]$. Then $\mathcal{S}$ is called a Riemann sum for f with respect to the partition (x_i) iff there exist $s_1, \ldots, s_n$ with $s_i \in [x_{i-1}, x_i]$ and

$$\mathcal{S} = \sum_{i=1}^{n} s_i \, \Delta x_i.$$

Definition 2.2 (Mesh of a partition)**.** Let $a \leq b$ and $(x_i)_{i=0}^n$ be a partition of $[a, b]$. Then we define the mesh of (x_i) as

$$\mu\big((x_i)\big) := \max_{1 \leq i \leq n} \Delta x_i.$$

Theorem 2.3. *Let $a \leq b$ and $f\colon [a, b] \to S$ be bounded and $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for any partition P of $[a, b]$, we have that*

$$\mu(P) < \delta \implies \underline{\int_a^b} f - L(f, P),\ U(f, P) - \overline{\int_a^b} f < \varepsilon,$$

and hence, if $\mathcal{S}$ is any Riemann sum corresponding to P, then

$$\mu(P) < \delta \implies \underline{\int_a^b} f - \varepsilon < \mathcal{S} < \overline{\int_a^b} f + \varepsilon.$$

Corollary 2.4. *Let $a \leq b$ and $f\colon [a, b] \to S$ be bounded. Let (P_i) be a sequence of partitions of $[a, b]$ such that $\mu(P_i) \to 0$. Then we have that*

$$L(f, P_i) \to \underline{\int_a^b} f \text{ and } U(f, P_i) \to \overline{\int_a^b} f.$$

Corollary 2.5 (Riemann sums converge to the integral)**.** *Let $a \leq b$ and $f\colon [a,b] \to S$ be integrable. Let (P_i) be a sequence of partitions of $[a,b]$ such that $\mu(P_i) \to 0$. Let $(\mathcal{S}_i)$ be a sequence of Riemann sums such that $\mathcal{S}_i$ is a Riemann sum of f with respect to the partition P_i. Then*

$$\mathcal{S}_i \to \int_a^b f.$$

Corollary 2.6 (The classical Riemann sums)**.** *Let $a \leq b$ and $f\colon [a,b] \to S$ be integrable. Then*

$$\frac{1}{n}\sum_{i=1}^{n} f\Big(a + i\,\frac{b-a}{n}\Big) \to \int_a^b f.$$

3 Properties of the integral

November 19, 2022

Theorem 3.1 (Limit of integrals)**.** *Let $a \leq b$ and $f\colon [a,b] \to S$ be bounded. Let $a_i \to a$ and $b_i \to b$ such that $a \leq a_i \leq b_i \leq b$ for each i. Let f be Riemann integrable on each $[a_i, b_i]$. Then f is Riemann integrable on $[a,b]$ with*

$$\int_{a_i}^{b_i} f \to \int_a^b f.$$

Theorem 3.2 (Domain additivity)**.** *Let $a \leq b$ and $f\colon [a,b] \to S$ be bounded. Let $a \leq x \leq b$. Then*

$$\underline{\int_a^b} f = \underline{\int_a^x} f + \underline{\int_x^b} f, \text{ and}$$
$$\overline{\int_a^b} f = \overline{\int_a^x} f + \overline{\int_x^b} f.$$

Further, we have that f is Riemann integrable on $[a,b]$ $\iff$ f is Riemann integrable on $[a,x]$ and $[x,c]$, in which case,

$$\int_a^b f = \int_a^x f + \int_x^b f.$$

Proposition 3.3. *Let $a \leq b$ and $f\colon [a,b] \to S$ be Riemann integrable. Let $x, y, z \in [a,b]$. Then f is integrable on all the sub-intervals determined by a, x, y, z, b and the* *following hold:*

$$\int_x^z f = \int_x^y f + \int_y^z f$$
$$\int_y^x f = -\int_x^y f$$

Theorem 3.4. *A continuous function is Riemann integrable on a closed bounded interval.*

Theorem 3.5. *A monotonic function is Riemann integrable on a closed bounded interval.*

Corollary 3.6. *Let $a \leq b$ and $f\colon [a,b] \to S$. Let $(x_i)_{i=0}^n$ be a partition of $[a,b]$ such that f is either continuous or monotone on each $[x_{i-1}, x_i]$. Then f is Riemann integrable on $[a,b]$ with*

$$\int_a^b f = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f.$$

Remark. *When we say "$f\colon S \to T$ is bounded away from zero", we mean that there is a $\delta > 0$ such that $|f(x)| \geq \delta$ for each $x \in S$.*

Notation. *For $f, g\colon S \to T$, we will denote the functions on $S \to \mathbb{R}$ given by $x \mapsto f(x)g(x), |f(x)|$ by fg and $|f|$ respectively.*

Theorem 3.7 (Integral manipulations)**.** *Let $a \leq b$ and $f, g\colon [a,b] \to S$ be bounded. Then the following hold:*

(i) $\underline{\int}_a^b (f+g) \geq \underline{\int}_a^b f + \underline{\int}_a^b g$.

(ii) $\overline{\int}_a^b (f+g) \leq \overline{\int}_a^b f + \overline{\int}_a^b g$.

(iii) $\underline{\int}_a^b (\alpha f) = \alpha \underline{\int}_a^b f$ *for* $\alpha \geq 0$.

(iv) $\overline{\int}_a^b (\alpha f) = \alpha \overline{\int}_a^b f$ *for* $\alpha \geq 0$.

Further, if f, g are Riemann integrable and $\alpha \in \mathbb{R}$, then the following hold:

(i) $f+g$, αf, fg *Riemann integrable as well with*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g, \text{ and}$$

$$\int_a^b (\alpha f) = \alpha \int_a^b f.$$

If g is, in addition, bounded away from zero, then f/g is also Riemann integrable.

(ii) If $f(x) \le g(x)$ for each $x \in [a,b]$, then

$$\int_a^b f \le \int_a^b g.$$

We also have that $|f|$ is Riemann integrable with

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

Theorem 3.8. *The composition of a continuous and a Riemann-integrable function is Riemann-integrable.*

Proposition 3.9. *Let*[3] *$a < b$ and $f\colon [a,b] \to \mathbb{R}$ be continuous with each $f(x) \ge 0$. Then*

$$\int_a^b f = 0 \implies f(x) = 0 \text{ for each } x.$$

Proposition 3.10 (Integral mean value)**.** *Let $a \le b$ and $f, g\colon [a,b] \to \mathbb{R}$ such that f is continuous and g is integrable with $g(x) \ge 0$ for each x. Then there exists a $c \in [a,b]$ such that*

$$\int_a^b fg = f(c) \int_a^b g.$$

4 The fundamental theorems of calculus

November 20, 2022

Theorem 4.1 ("Integral of the derivative")**.** *Let $a \le b$ and $F\colon [a,b] \to S$ be continuous, and differentiable on (a,b). Let $f\colon [a,b] \to T$ be Riemann integrable with $f(x) = F'(x)$ for each $x \in (a,b)$. Then*

$$\int_a^b f = F(b) - F(a).$$

[3] This doesn't hold for $a = b$.

Theorem 4.2 ("Derivative of the integral"). *Let $a \leq b$ and $f\colon [a,b] \to S$ be Riemann integrable. Let $F\colon [a,b] \to T$ such that $F(x) = \int_a^x f$ for each[4] $x \in [a,b]$. Then the following hold:*

(i) F is Lipschitz continuous.

(ii) f is continuous at $c \in [a,b] \implies F$ is differentiable at c with

$$F'(c) = f(c).$$

Corollary 4.3 (Transfer of properties from f to F). *Let $a \leq b$ and $f\colon [a,b] \to S$ be continuous. Define $F\colon [a,b] \to \mathbb{R}$ by $x \mapsto \int_a^x f$. Then the following hold:*

(i) F is monotonically increasing $\iff f(x) \geq 0$ for each x.

(ii) F is convex $\iff f$ is monotonically increasing.

Result 4.4. For every $n \geq 0$, there exists a function $f\colon [-1,1] \to \mathbb{R}$ such that $\operatorname{dom} f^{(n)} = [-1,1]$, but $\operatorname{dom} f^{(n+1)} = [-1,1] \setminus \{0\}$.

Proposition 4.5. *Let $a \leq b$ and $F\colon [a,b] \to S$. Then the following are equivalent:*

(i) F is continuously differentiable on $[a,b]$.

(ii) There exists a continuous function $f\colon [a,b] \to \mathbb{R}$ such that for each $x \in [a,b]$, we have that

$$F(x) = F(a) + \int_a^x f.$$

Theorem 4.6 (Change of variables). *Let $\alpha \leq \beta$ be reals and $\phi\colon [\alpha,\beta] \to S$ be differentiable with ϕ' being Riemann integrable, and $\phi([\alpha,\beta]) \subseteq [a,b]$ where $a \leq b$. Let $f\colon [a,b] \to T$ be continuous. Then $(f \circ \phi)\phi'\colon [a,b] \to \mathbb{R}$ is Riemann integrable with*

$$\int_{\phi(\alpha)}^{\phi(\beta)} f = \int_\alpha^\beta (f \circ \phi)\phi'.$$

Notation. *$\int_a^b f(t)\,\mathrm{d}t$ will stand for $\int_a^b f$. In a similar spirit, $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$ will stand for f'.*

These are useful when we want to write the "rule" of f instead of f, and then x denotes what is the "variable" in that rule.

[4] Theorem 3.2 ensures that f is integrable on each $[a,x]$.

Proposition 4.7 (A version of Leibniz)**.** *Let $\alpha \leq \beta$ be reals and $a \leq b$. Let $u, v\colon [\alpha, \beta] \to S$ be differentiable with u', v' Riemann integrable and their ranges lying inside $[\alpha, \beta]$. Let $f\colon [a, b] \to T$ be continuous. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{u(x)}^{v(x)} f = (f \circ v)v' - (f \circ u)u'.$$

Theorem 4.8 (Integration by parts)**.** *Let $a \leq b$ and $f\colon [a, b] \to S$ be differentiable with f' being Riemann integrable, and $g\colon [a, b] \to T$ be continuous. Let $G\colon [a, b] \to T$ be such that $G(x) = \int_a^x g$. Then $fg, f'G\colon [a, b] \to \mathbb{R}$ are Riemann integrable with*[5]

$$\int_a^b fg = f(b)G(b) - f(a)G(a) - \int_a^b f'G.$$

Theorem 4.9 (Another version of Taylor)**.** *Let $a \leq b$ and $f\colon [a, b] \to S$ be $n + 1$ times continuously differentiable for an $n \geq 0$. Then*[6]

$$f(b) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(b - a)^i + \frac{1}{n!} \int_a^b (b - t)^n f^{(n+1)}(t) \, \mathrm{d}t.$$

[5] Note that $G(a) = 0$, but I'm writing it for the sake of beauty.

[6] In the integrand, the function is on $[a, b] \to \mathbb{R}$.