Contents

Chapter I

Quotient topology

Convention. *For the rest of the document, unless stated otherwise:*

(i) *Sets are topological spaces.*

(ii) *Functions between topological spaces are continuous.*

(iii) $I = [0, 1]$.

(iv) B^n, D^n *will denote the unit open ball and disc in* $\mathbb{R}^n$ *under the* l_2-*norm.*

(v) S^n *will denote the unit sphere in* $\mathbb{R}^{n+1}$ *under the* l_2-*norm.*

(vi) *Depending on the context,* S^1 *may also mean the unit circle in* $\mathbb{C}$*, which is homeomorphic to the unit circle in* $\mathbb{R}^2$.

1 Saturations and saturated sets

September 19, 2023

Definition 1.1 (Saturations and saturated sets)**.** Let $f\colon X \to Y$ be a set theoretic function. Then we define the *saturation* of an $A \subseteq X$ to be $f^{-1}(f(A))$.

Further, A is called *saturated* iff A equals its saturation.

Lemma 1.2 (Characterizing saturations and saturated sets)**.** *Let* $f\colon X \to Y$ *be set theoretic and* $A \subseteq X$*. Then the following hold:*

(i) *The saturation of* A *is the smallest saturated set containing* A*.*

(ii) *The following are equivalent:*

(a) A *is the inverse image of some subset of* B*.*

(b) $f^{-1}(\{y\})$ *lies in either* A *or* $X \setminus A$*, for each* $y \in Y$*.*

(c) A is saturated.

(iii) If A is saturated, then so is $X \setminus A$ with $f(X \setminus A) = f(X) \setminus f(A)$.

2 Quotient topology

September 19, 2023

Lemma 2.1 (Terminal topologies induced by a function)**.** *Let $f \colon X \to Y$ be set theoretic. Then the following hold:*

(i) If X is a topological space, then

$$\{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$$

is the largest topology on Y that makes f continuous.

(ii) If Y is a topological space, then

$$\{f^{-1}(V) \subseteq X : V \text{ is open in } Y\}$$

is the smallest topology on X that makes f continuous.

Definition 2.2 (Quotient topology)**.** Given an equivalence relation $\sim$ on X, the largest topology on $X/{\sim}$ that makes the canonical function $X \to X/{\sim}$ continuous is called the quotient topology on $X/{\sim}$.

Remark. *The map $X \to X/{\sim}$ is the model for quotient maps.*

Definition 2.3 (Quotient maps)**.** A continuous surjection $p \colon X \to Y$ such that V is open in Y whenever $p^{-1}(V)$ is open in X, is called a quotient map.

Corollary 2.4.

(i) Quotients preserve compactness.

(ii) Composition of quotients is quotient.

Example 2.5 (Restrictions of quotients needn't be quotient!)**.** Consider the restriction of the projection $\mathbb{R}^2 \to \mathbb{R}$ onto the first coordinate, to the subspace $\{(0,0)\} \cup \{(x,y) : y = 1 \text{ and } x \neq 0\}$.

Example 2.6 (Quotients needn't preserve local compactness)**.** $\mathbb{R}$ with integers identified together is Hausdorff, but not locally compact.

Lemma 2.7 (Characterizing the "quotientness" condition)**.** *Let $p\colon X \to Y$, not necessarily continuous, be a surjection. Then the following are equivalent:*[1]

(i) $p^{-1}(V)$ *open* $\implies$ V *open.*

(ii) p *maps saturated opens to opens.*

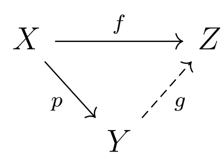
(iii) p *maps saturated closeds to closeds.*

(iv) $p^{-1}(K)$ *is closed* $\implies$ K *is closed.*

Corollary 2.8. *A continuous open or closed map*[2] *is quotient.*

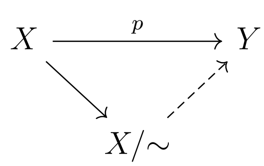
Example 2.9 (Quotient maps needn't be open or closed)**.** Consider the restriction of the projection $\mathbb{R}^2 \to \mathbb{R}$ onto the first coordinate, to the subspace $\{(x, y) : x \geq 0$ or $y = 0\}$. Then this is a quotient map which is neither closed nor open.

Proposition 2.10 (Universal property of quotient maps)**.** *Let $p\colon X \to Y$ be a quotient map. Let $f\colon X \to Z$ factor through p:*



Then f is continuous $\iff$ g is.

Lemma 2.11 ({quotient maps} $\leftrightarrow$ {quotient spaces})**.** *Let $p\colon X \to Y$ be quotient. Let $\sim$ be the equivalence relation on X induced by f. Then the factor map $X/{\sim} \to Y$ is a homeomorphism:*



[1] Surjectivity of p is needed for "(ii) $\Rightarrow$ (iii)" and "(iii) $\Rightarrow$ (iv)".

[2] Note that following the conventions, a map is by default continuous.

3 Products of quotient maps

November 14, 2023

Theorem 3.1 (Whitehead)**.** *If $p\colon X \to Y$ is a quotient map and Z is locally compact Hausdorff, then $p \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is also quotient.*

Example 3.2 (Necessity of local compactness)**.** Let $\sim$ be the equivalence relation on $\mathbb{R}$ identifying integers together, and let $p\colon \mathbb{R} \to \mathbb{R}/{\sim}$ be the corresponding quotient map. Then $p \times \mathrm{id}_{\mathbb{Q}}\colon \mathbb{R} \times \mathbb{Q} \to (\mathbb{R}/{\sim}) \times \mathbb{Q}$ is not quotient.

Corollary 3.3. *If $p\colon X \to Y$ and $q\colon Z \to W$ are quotient maps with X, W (or Y, Z) being locally compact Hausdorff, then $p \times q$ is also quotient.*

4 Some examples

September 19, 2023

Proposition 4.1 (Circle "wrapped onto itself n times" is circle)**.** *Let $n \geq 1$. Then S^1 under the quotient topology due to the equivalence relation due to f, is homeomorphic to S^1.*

Notation. *For $A \subseteq X$, when there's no chance of confusion,[3] we'll write X/A for the quotient space obtained by identifying the points A together in X.*

Proposition 4.2. *$D^n/S^{n-1} \cong S^n$.*

4.1 Quotients of $I \times I$

September 20, 2023

Remark. *While defining equivalence relations[4], we'll omit mentioning that the equal points are related. We'll also omit mentioning $y \sim x$ if we have mentioned $x \sim y$.*

[3] For instance, A might be a subgroup of X.

[4] Actually that a relation is an equivalence relation must be checked after we have defined it in the first place, but still...

Proposition 4.3 (Constructing spaces out of $I \times I$)**.** *Let $X := I \times I$. Define the following equivalence relations on X:*

(i) $(x, y) \sim_1 (x, y')$ *and* $(0, y) \sim_1 (1, y)$.

(ii) $(0, y) \sim_2 (1, y)$.

(iii) $(x, 0) \sim_3 (x, 1)$ *and* $(0, y) \sim_3 (1, y)$.

Then we have:

(i) $X/{\sim_1} \cong S^1$.

(ii) $X/{\sim_2} \cong S^1 \times I$.

(iii) $X/{\sim_3} \cong S^1 \times S^1$.

4.2 Cones and suspensions

September 20, 2023

Definition 4.4 (Cones and suspensions)**.** The *cone* of a space X is $X \times I$ with points of $X \times \{1\}$ identified, whereas the *suspension* of X is $X \times I$ with points of $X \times \{0\}$ and $X \times \{1\}$ respectively identified.

Proposition 4.5.

(i) Cone of S^{n-1} *is* D^n.

(ii) Suspension of S^n *is* S^{n+1}.

4.3 Wedge products

November 12, 2023

Definition 4.6 (Wedge sum of topological spaces)**.** Given pointed spaces (X_α, x_α), we define their wedge product $\bigvee_\alpha (X_\alpha, x_\alpha)$ to be the quotient of $\bigsqcup_\alpha X_\alpha$ by identifying all the (x_α, α)'s together.

Proposition 4.7 (Hawaiian earring $\not\cong \bigvee_{n=1}^\infty$ (circle))**.** *For $n \geq 1$, let C_n be the circle of radius $1/n$ in $\mathbb{R}^2$ centered at $(1/n, 0)$. Let $X := \bigcup_n C_n$ and*

$$\mathcal{H} := \bigvee_n \bigl(C_n, (0, 0)\bigr).$$

Then X is compact, whereas the sequence of points $(1/n, 0)$ has no convergent subsequence in $\mathcal{H}$.

5 Quotients and Hausdorffness

September 19, 2023

Example 5.1 (Quotients don't preserve Hausdorffness).

(i) For E is dense in X, any nonempty open set in X/E contains E.

(ii) The group quotient $\mathbb{R}/\mathbb{Q}$ is indiscrete.

(iii) *(The real line with two origins)*. Let $X := \{(x, y) : y = 0$ or $y = 1\}$ and define an equivalence relation on X by $(x, 0) \sim (x, 1)$ for $x \neq 0$. Then we can't separate $\{(0, 0)\}$ and $\{(0, 1)\}$ via opens in $X/\sim$.

Theorem 5.2 (The Hausdorff criterion). *Let $p: X \to Y$ be closed, continuous and surjective. Let X be normal*[5] *with singletons closed. Then Y is Hausdorff.*

Proposition 5.3 (Comparing with the openness and closedness of the induced relation). *Let $p: X \to Y$ not necessarily be continuous. Set $R := \{(x, y) \in X \times X : p(x) = p(y)\}$. Then the following hold:*

(i) p is quotient and R is open $\implies$ Y is discrete $\implies$ p is open.

(ii) p continuous and Y is Hausdorff $\implies$ R is closed.

(iii) p open and surjective, and R is closed $\implies$ Y is Hausdorff.

(iv) If p is quotient and X is compact Hausdorff, then R is closed $\iff$ Y is Hausdorff.

Example 5.4 (Counters to converses).

(i) *(p open $\not\Rightarrow$ R open; R closed $\not\Rightarrow$ p closed)*. Take p to be the projection $\mathbb{R}^2 \to \mathbb{R}$ onto the first coordinate.

(ii) *(p closed $\not\Rightarrow$ R closed)*. Take p to be the identity map on any non-Hausdorff X.

6 Projective spaces

September 19, 2023

Definition 6.1 (Projective spaces). Let $n \geq 0$ and define an equivalence relation on S^n by $x \sim -x$. Then we define $\mathbb{R}P^n$ to be the quotient space $S^n/\sim$.

[5] That is, closeds can be separated via opens.

Proposition 6.2. *Each $\mathbb{R}P^n$ is compact, (path) connected and Hausdorff.*

Theorem 6.3 (Different descriptions of $\mathbb{R}P^2$)**.** *For $n \geq 1$, define the following equivalence relations:*

(i) On $\mathbb{R}^n \setminus \{0\}$: $x \sim_1 \lambda x$ for $\lambda \neq 0$.

(ii) On D^n: $x \sim_2 -x$ for $x \in S^{n-1}$.

Then the quotient spaces $(\mathbb{R}^n \setminus \{0\})/\sim_1$ and $D^n/\sim_2$ are both homeomorphic to $\mathbb{R}P^n$.

Chapter II

Homotopy

Convention. *Throughout the rest of the document, unless stated otherwise:*

(i) For $x_0 \in X$, the constant function $x \mapsto x_0$ on either $X \to X$ or $I \to X$ (depending on the context) will be denoted by c_{x_0}.

(ii) Statements involving $\mathbb{K}$ will mean two statements, one for $\mathbb{R}$ and one for $\mathbb{C}$.

1 Relative homotopies

September 20, 2023

Definition 1.1 (Relative homotopy)**.** Let $f, g\colon X \to Y$ be continuous and $A \subseteq X$. Then a *homotopy from f to g relative to A* is a continuous map $F\colon X \times I \to Y$ such that the following hold:

(i) $F(x, 0) = f(x)$.

(ii) $F(x, 1) = g(x)$.

(iii) $F(a, t)$ is independent of t for all $a \in A$.

We say f and g are *homotopic relative to A* iff there exists a homotopy between them relative to A.

If $A = \emptyset$, then we omit "relative to A".

Definition 1.2 (Nullhomotopic maps)**.** A map is called nullhomotopic iff it is homotopic to some constant map.

Example 1.3 (Straight line homotopy)**.** Let Y be a convex set of a topological vector space over $\mathbb{K}$. Then any two $f, g\colon X \to Y$ are homotopic relative to the equalizer of f, g via $F\colon X \times I \to Y$ given by

$$(x,t) \mapsto (1-t)\, f(x) + t\, g(x).$$

Example 1.4 (Homotopies on S^n)**.** Any $f, g\colon X \to S^n$ for which $f(x) \neq -g(x)$[1] for all $x \in X$, are homotopic via

$$(x,t) \mapsto \frac{(1-t)\, f(x) + t\, g(x)}{\|(1-t)\, f(x) + t\, g(x)\|}.$$

Thus, for any $f\colon S^n \to S^n$, the following hold:

(i) If f has no fixed points, then f is homotopic to the antipodal map.

(ii) If $f(x) \neq -x$ for any $x \in X$, then f is homotopic to the identity map.

Example 1.5 (Motivation for hairy ball)**.**

(i) *(Normal vector fields allow to deform* id *into* ap*)***.** Let $v\colon S^n \to S^n$ be continuous with $v(x) \perp x$ for each $x \in S^n$. Then

$$(x,t) \mapsto (\cos \pi t)\, x + (\sin \pi t)\, v(x)$$

defines a homotopy from identity to the antipodal map on S^n.

(ii) *(Normal vector fields exist on S^n for n odd)***.** For n odd, the following defines a continuous vector field $S^n \to S^n$ that is normal to S^n at each point:

$$(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \mapsto (-x_2, x_1, \ldots, -x_{2n}, x_{2n-1})$$

(iii) On S^n for n odd, identity is homotopic to the antipodal map.

Proposition 1.6. *For a fixed subspace A of X, "being homotopic relative to A" is an equivalence relation on the set of all continuous $X \to Y$.*

Proposition 1.7 (RelHTop)**.** *The pairs (X, A) of spaces X and their subspaces A form a category wherein the morphisms from (X, A) to (Y, B) are the continuous*

[1] To make it work for complex S^n, we must have $f(x) \neq e^{i\theta} g(x)$ for $\theta \in (0, 2\pi)$.

$f\colon X \to Y$ with $f(A) \subseteq B$, modded out by "being homotopic relative to A".[2] *The composition of $[f]_{A,B}\colon (X,A) \to (Y,B)$ and $[g]_{B,C}\colon (Y,B) \to (Z,C)$ is given by*[3]

$$[g]_{B,C}\,[f]_{A,B} = [g \circ f]_{A,C}.$$

In this category, the identity morphism on (X,A) is

$$[\mathrm{id}_X]_{A,A}.$$

Definition 1.8 (Relative homotopic equivalence)**.** In $\mathsf{RelHTop}$, isomorphisms are called *relative homotopic equivalences*, and isomorphic objects are said to be *relatively homotopically equivalent* or of *same relative homotopic type*. As before, if the subspace is empty for both the pairs, then we drop "relatively".

Remark. *Sometimes, we'll write "$f\colon X \to Y$ is a homotopic equivalence" to mean that $[f]_{\emptyset,\emptyset}\colon (X,\emptyset) \to (Y,\emptyset)$ is a homotopy equivalence.*

Corollary 1.9 (Alternate way of expressing various things in $\mathsf{RelHTop}$)**.** *Let $A \subseteq X$ and $B \subseteq Y$. Then the following hold:*

(i) For $f, g\colon X \to Y$ with $f(A), g(A) \subseteq B$, we have $[f]_{A,B} = [g]_{A,B} \iff f$ is homotopic to g relative to A.

(ii) (X,A) is homotopically equivalent to $(Y,B) \iff$ there exist $f\colon X \to Y$ and $g\colon Y \to X$ such that the following hold:

  (a) $f(A) \subseteq B$ and $g(B) \subseteq A$.

  (b) $g \circ f$ is homotopic to id_X relative to A.

  (c) $f \circ g$ is homotopic to id_Y relative to B.

Proposition 1.10. *Let X be (path) connected, and $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ is homotopic to id_Y. Then Y is also (path) connected.*

Corollary 1.11. *Homotopy equivalences preserve (path) connectivity.*

[2] To be completely precise, the morphisms should also contain the information of their domain and codomain objects.

[3] The subscript A and B denote the fact that the equivalence relations are different.

2 Contractible spaces

September 20, 2023

Definition 2.1 ((Relatively) contractible spaces)**.** A space X is called

(i) *contractible to* $x_0 \in X$ iff X is homotopically equivalent to $\{x_0\}$; and,

(ii) *contractible relative to* $x_0 \in X$ iff $(X, \{x_0\})$ is relatively homotopically equivalent to $(\{x_0\}, \{x_0\})$.

We say that X is *contractible (relatively)* iff X is contractible (relative) to some $x_0 \in X$.

Corollary 2.2.

(i) Contractibility is preserved under homotopic equivalence.[4]

(ii) Contractible spaces are path-connected.[5]

(iii) (Characterizing (relative) contractibility)**.** *X is contractible (relative) to $x_0 \in X \iff$ the constant map c_{x_0} is homotopic to id_X (relative to $\{x_0\}$).*

(iv) A contractible space is contractible to all of its points.

(v) Cones are contractible.

(vi) If either X or Y is contractible, then any map $X \to Y$ is nullhomotopic.

(vii) If X is contractible and Y path-connected, then any two continuous maps $X \to Y$ are homotopic.

(viii) Products and retracts of contractible spaces are contractible.

Proposition 2.3. *If X is contractible relative to x_0, then X is weakly locally path connected at x_0.*

Remark. *Proposition 2.3 can't be strengthened by either of the following ways:*

(i) Dropping "relative": Consider Hatcher's zigzg space.

(ii) Having "locally path connected": Consider iterated broom.

Example 2.4 (Comb space can't be contracted relatively to $(0,1)$)**.** Consider the following subspace of $\mathbb{R}^2$:

$$\mathcal{C} := \left(\overline{\{1/n : n \geq 1\} \times I}\right) \cup \left(I \times \{0\}\right)$$

Then $\mathcal{C}$ is not weakly locally path connected at $(0,1)$ so that it's not contractible relative to $(0,1)$. However, it can be contracted relative to $(0,0)$.

[4] This is not true of relative contractibility. Consider comb space of Example 2.4.

[5] Converse not true! Consider S^1. See Corollary 4.2.

3 Retracts

September 20, 2023

Definition 3.1 (((Strong) deformation) retracts)**.** Let $A \subseteq X$. Then a continuous $r\colon X \to A$ is called

(i) a *retract* iff $r \circ \iota = \mathrm{id}_A$;

(ii) a *deformation retract* iff $r \circ \iota = \mathrm{id}_A$ and $\iota \circ r$ is homotopic to id_X; and,

(iii) a *strong deformation retract* iff $r \circ \iota = \mathrm{id}_A$ and $\iota \circ r$ is homotopic to id_X relative to A.

Accordingly, we call A a ((strong) deformation) retract of X.

Corollary 3.2. *(Strong) deformation retracts are (relative) homotopy equivalences.*

Example 3.3 (Retract needn't be a homotopy equivalence)**.** A point of a non-path-connected space is not homotopically equivalent to the space.[6]

Corollary 3.4 (Retractibility and contractibility to a point)**.** *Let $x_0 \in X$. Then the following hold:*

(i) $\{x_0\}$ is a retract.[7]

(ii) $\{x_0\}$ is a deformation retract of $X \iff X$ is contractible to x_0.

(iii) $\{x_0\}$ is a strong deformation retract of $X \iff X$ is contractible relative to x_0.

Example 3.5 (Not every subspace is a retract)**.** $\{0,1\}$ is not a retract of I.

Example 3.6 (Retract $\not\Rightarrow$ deformation retract $\not\Rightarrow$ strong deformation retract)**.**

(i) A point can't be a deformation retract of a non-path-connected space.

(ii) $\{(0,1)\}$ of comb space is a deformation retract but not strongly.

Example 3.7. S^n is a strong deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$ via $r\colon x \mapsto x/\|x\|$.

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[6] See Corollary 3.4.

[7] Note that the only candidate for the retract map is the constant map $X \to \{x_0\}$.

4 The fundamental groupoid

September 20, 2023

Notation. *We'll use these notations:* $\mathrm{Path}(X; x, y)$ *and* $\mathrm{Loop}(X; x)$.

Proposition 4.1 (Operations on paths)**.** *For a space* X*, let* $\alpha \in \mathrm{Path}(X; x, y)$*, and* $\beta \in \mathrm{Path}(X; y, z)$*. Then there exist the following paths:*

(i) (Join of α and β)**.** *A path* $\alpha * \beta \in \mathrm{Path}(X; x, z)$ *such that*

$$t \mapsto \begin{cases} \alpha(2t), & t \in [0, 1/2] \\ \beta(2t-1), & t \in [1/2, 1] \end{cases}.$$

(ii) (Inverse of α)**.** *A path* $\alpha^{-1} \in \mathrm{Loop}(X; y, x)$[8] *such that*

$$t \mapsto \alpha(1-t).$$

Remark.

(i) Join of paths is not associative.

(ii) α^{-1} *is just* $\alpha \circ f$ *where* $f \colon I \to I$ *is given by* $t \mapsto 1 - t$.

Definition 4.2 (Path homotopy)**.** A homotopy between two paths in a space, relative to $\{0, 1\}$ is called a *path homotopy* between them. We similarly define *path homotopic* paths.

Corollary 4.3. *"Being path homotopic" is an equivalence relation on* $\mathrm{Path}(X; x, y)$ *for all* $x, y \in X$.

Lemma 4.4. *Let* α *be a path from* x *to* y *in* X *and* $f \colon I \to I$ *be continuous with* $f(0) = 0$ *and* $f(1) = 1$*. Then* α *is path homotopic to* $\alpha \circ f$.

Proposition 4.5 (The fundamental groupoid)**.** *The points of a space* X *form a category* $\Pi(X)$ *with morphisms from* x *to* y *being paths from* x *to* y *modded out by "being path homotopic". The composition of* $[\alpha] \colon x \to y$ *and* $[\beta] \colon y \to z$ *is given by*[9]

$$[\beta]\,[\alpha] = [\alpha * \beta].$$

The identity morphism on x *is* $[c_x]$*. Further,* $\Pi(X)$ *forms a groupoid with*

$$[\alpha]^{-1} = [\alpha^{-1}].$$

[8] Of course, the notation α^{-1} is not great.

[9] Note that $[\alpha]$ and $[\beta]$ are classes of *different* equivalence relations.

Proposition 4.6 (The functor $\mathsf{Top} \to \mathsf{Gpd}$[10])**.** *The following defines a functor* $\mathsf{Top} \to \mathsf{Gpd}$*:*

$$\begin{array}{ccc} X & & \Pi(X) \\ {\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle \Pi(f)} \\ Y & & \Pi(Y) \end{array}$$

where $\Pi(f)\colon \Pi(X) \to \Pi(Y)$ *is the functor given by*

$$\begin{array}{ccc} x & & f(x) \\ {\scriptstyle [\alpha]}\downarrow & \overset{\Pi(f)}{\longmapsto} & \downarrow{\scriptstyle [f\circ\alpha]} \\ y & & f(y) \end{array}.$$

5 The fundamental group

September 21, 2023

Definition 5.1 (The fundamental group)**.** The fundamental group $\pi_1(X, x)$ of a space X at $x \in X$ is the group of morphisms associated with the full subcategory of $\Pi(X)$ generated by the single object x.

Corollary 5.2 (The functor $\mathsf{pTop} \to \mathsf{Grp}$)**.** *The following defines a functor* $\mathsf{pTop} \to \mathsf{Grp}$*:*

$$\begin{array}{ccc} (X, x) & & \pi_1(X, x) \\ {\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle f_*} \\ (Y, y) & & \pi_1(Y, y) \end{array}$$

where f_* *is given by*

$$f_*([\alpha]) = [f \circ \alpha].$$

Proposition 5.3 (The functor $\Pi(X) \to \mathsf{Grp}$)**.** *For a fixed space* X*, the following defines a functor* $\Pi(X) \to \mathsf{Grp}$*:*

$$\begin{array}{ccc} x & & \pi_1(X, x) \\ {\scriptstyle [\gamma]}\downarrow & \longmapsto & \downarrow{\scriptstyle \phi_{[\gamma]}} \\ y & & \pi_1(X, y) \end{array}$$

[10] Gpd is the full subcategory of Cat comprising of groupoids.

where $\phi_{[\gamma]}$ is given by

$$\phi_{[\gamma]}([\alpha]) = [\gamma]\,[\alpha]\,[\gamma]^{-1}.$$

Further, $\pi_1(X, x)$ is abelian $\iff$ for all points y we have

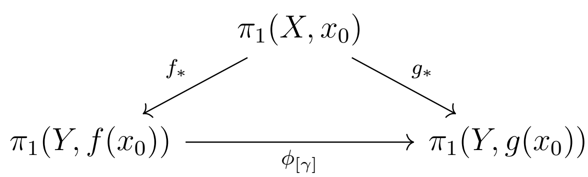
$$\phi_{[\gamma_1]} = \phi_{[\gamma_2]}$$

for all $[\gamma_1], [\gamma_2] \colon x \to y$.

Corollary 5.4. *If $\gamma \in \mathrm{Path}(X; x, y)$, then $\pi_1(X, x) \cong \pi_2(X, y)$ via $\phi_{[\gamma]}$. Thus, $\pi_1(X, x)$ is independent of x if X is path connected.*

Notation. *Thus, we'll use $\pi_1(X)$ for path-connected X when we just want to focus on $\pi_1(X, x)$ up to group isomorphisms.*

Proposition 5.5. *Let $f, g \colon X \to Y$ be homotopic via H and $x_0 \in X$. Define $\gamma \in \mathrm{Path}(Y; f(x_0), g(x_0))$ by $t \mapsto H(x_0, t)$. Then the following diagram commutes:*



Corollary 5.6 (Fundamental groups are preserved under homotopy equivalence)**.** *Let $f \colon X \to Y$ be a homotopic equivalence*[11] *and $x \in X$. Then $\pi_1(X, x) \cong \pi_1(Y, f(x))$.*

Corollary 5.7. *The fundamental group of a contractible space is trivial.*

6 $\pi_1(S^n)$ for $n \geq 2$

November 14, 2023

Remark. *We'll compute $\pi_1(S^1)$ in Corollary 4.2 of Chapter III.*

Definition 6.1 (Simply connected spaces)**.** A space X is said to be simply connected iff it is path-connected with $\pi_1(X) = 0$.

[11] See remark.

Proposition 6.2 (A version of van Kampen)**.** *If X can be written as a union of two simply connected open subspaces whose intersection is nonempty and path-connected, then X is simply connected.*

Proposition 6.3 (Stereographic projections)**.** *Let $n \geq 1$. Then the functions $f\colon S^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to S^n \setminus \{e_{n+1}\}$ given by*

$$f(x_1, \dots, x_{n+1}) := \Big(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\Big)$$
$$g(y_1, \dots, y_n) := \Big(\frac{2y_1}{1 + \|y\|^2}, \dots, \frac{2y_n}{1 + \|y\|^2}, \frac{-1 + \|y\|^2}{1 + \|y\|^2}\Big)$$

define homeomorphisms which are inverses of each other.

Corollary 6.4.

(i) S^n is simply connected for $n \geq 2$.

(ii) Any non-surjective map $X \to S^n$ is nullhomotopic.

7 Miscellaneous

November 9, 2023

Proposition 7.1 (Borsuk-Ulam versions)**.** *For $n \geq 1$, the following are equivalent:*

(i) Every continuous map $S^n \to \mathbb{R}^n$ has an antipodal pair on which f agrees.

(ii) Every continuous antipode-preserving map $S^n \to \mathbb{R}^n$ vanishes somewhere.

(iii) There is no continuous antipode-preserving map $S^n \to S^{n-1}$.

(iv) If $n + 1$ closed sets cover S^n, then one of them contains an antipodal pair.

Prove (iv)⇒(i)!

Corollary 7.2. *Borsuk-Ulam holds for $n = 1$.*

Chapter III

Covering spaces

1 Basic stuff

November 8, 2023

Definition 1.1 (Evenly covered)**.** Let $p\colon \tilde{X} \to X$ be continuous. Then a subset Y of X is said to be evenly covered by p iff $p^{-1}(Y)$ is a disjoint union of open sets, each homeomorphic to Y via p's respective restrictions.

Corollary 1.2.

(i) Open subsets which are subsets of evenly covered sets are evenly covered.

(ii) If the whole space X is evenly by $p\colon \tilde{X} \to X$, then $\tilde{X} \cong X \times F$ where F is a discrete space.[1]

Definition 1.3 (Covering projections)**.** A map $p\colon \tilde{X} \to X$ is called a *covering projection* of the *base space* X by the *covering space* $\tilde{X}$ iff each point in X has an evenly covered open neighborhood.

Corollary 1.4.

(i) Homeomorphisms are covering projections.

(ii) Covering projections are open.

(iii) Injective covering projections are homeomorphisms.

(iv) For a discrete space F, the projection $X \times F \to X$ is a covering projection.

(v) Any fibre of a covering projection is discrete.

(vi) Covering projections are local homeomorphisms.

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[1] Compare with (iv) of Corollary 1.4.

(vii) Restrictions of covering projections to saturated sets are covering projections.

(viii) Finite product of covering projections is a covering projection.

Proposition 1.5 (Some covering projections)**.** *The following are covering projections:*

(i) $\mathbb{R} \to S^1$ *given by* $t \mapsto e^{i2\pi t}$.

(ii) $S^1 \to S^1$ *given by* $z \mapsto z^n$ *for any* $n \geq 1$.

(iii) $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ *given by* $z \mapsto e^z$.

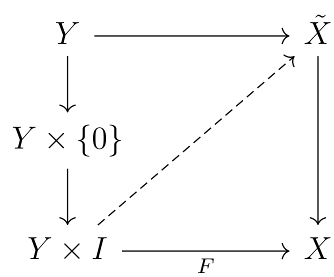
Example 1.6. The restriction of $t \mapsto e^{i2\pi t}$ to $(0, 2)$ is not a covering projection but nevertheless a local homeomorphism.

2 Lifting properties

November 9, 2023

Proposition 2.1 (A sufficient condition for covering-lifts to be unique)**.** *Let* $f : Y \to X$ *where* Y *is connected. Then any two lifts of* f *through a covering of* X *that agree on some point in* Y *are the same.*

Theorem 2.2 (Homotopy-lifting property of coverings)**.** *Let* $F : Y \times I \to X$ *a homotopy. Then any lift of* $y \mapsto F(y, 0)$ *through a covering of* X *extends uniquely to a lift of* F.



Corollary 2.3 (Paths and homotopies between them can be lifted through coverings)**.** *Let* $p : \tilde{X} \to X$ *be a covering. Then the following hold:*

(i) Let α *be a path in* X *starting at* x_0*. Then for any* $\tilde{x}_0 \in p^{-1}(\{x_0\})$*, there exists a unique path in* $\tilde{X}$ *starting at* $\tilde{x}_0$ *that lifts* α.

(ii) Let $F : I \times I \to X$ *be a homotopy. Set* $x_0 := F(0, 0)$ *and let* $\tilde{x}_0 \in p^{-1}(\{x_0\})$*. Then there exists a unique homotopy* $\tilde{F}$ *in* $\tilde{X}$ *with* $\tilde{F}(0, 0) = \tilde{x}_0$*, lifting* F.

Notation. *This allows to denote the lift in (i) by $\alpha^{\sim}_{\tilde{x}}$ (when the covering projection being talked of is clear from the context).*

Proposition 2.4 (Monodromy)**.** *Let $p\colon \tilde{X} \to X$ be a covering and $\tilde{\alpha}$, $\tilde{\beta}$ be paths in X starting at the same point. Then $\tilde{\alpha}$ is path homotopic to $\tilde{\beta}$ $\iff$ $p \circ \tilde{\alpha}$ is path homotopic to $p \circ \tilde{\beta}$.*

Corollary 2.5. *Homomorphisms between fundamental groups induced by covering projections are injective.*

3 The action of the fundamental group

Novemvber 14, 2023

Lemma 3.1 (Lifts of joins and inverses)**.** *Let $p\colon \tilde{X} \to X$ be a covering projection and $\alpha \in \mathrm{Path}(X; x, y)$, $\beta \in \mathrm{Path}(X; y, z)$. Let $\tilde{x} \in p^{-1}(\{x\})$. Set $\tilde{y} := \alpha^{\sim}_{\tilde{x}}(1)$. Then the following hold:*

(i) $\tilde{y} \in p^{-1}(\{y\})$.

(ii) $(\alpha * \beta)^{\sim}_{\tilde{x}} = \alpha^{\sim}_{\tilde{x}} * \beta^{\sim}_{\tilde{y}}$.

(iii) $(\alpha^{-1})^{\sim}_{\tilde{y}} = (\alpha^{\sim}_{\tilde{x}})^{-1}$.

Corollary 3.2 (The functor $\Pi(X) \to \mathsf{Set}$)**.** *Let $p\colon \tilde{X} \to X$ be a covering projection. Then the following defines a functor $\Pi(X) \to \mathsf{Set}$:*

$$\begin{array}{ccc} x & & p^{-1}(\{x\}) \\ {\scriptstyle [\alpha]}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \psi_{[\alpha]}} \\ y & & p^{-1}(\{y\}) \end{array}$$

where $\psi_{[\alpha]}$ is given by

$$\psi_{[\alpha]}(\tilde{x}) = \alpha^{\sim}_{\tilde{x}}(1).$$

Corollary 3.3. *For a covering projection, the fibres of path-connected points have the same cardinality.*

Corollary 3.4 (The action of $\pi_1(X, x)$ on $p^{-1}(\{x\})$)**.** *Let $p\colon \tilde{X} \to X$ be a covering projection and $x \in X$. Then $\pi_1(X, x)$ acts on $p^{-1}(\{x\})$ via*

$$[\alpha]\, \tilde{x} = \alpha^{\sim}_{\tilde{x}}(1).$$

Further, the following hold:

(i) $\operatorname{Stab}(\tilde{x}) = p_*\big(\pi_1(\tilde{X}, \tilde{x})\big)$.

(ii) *If $\tilde{X}$ is path-connected and $\tilde{x} \in p^{-1}(\{x\})$, then the following hold:*

- *(a) The action is transitive.*
- *(b)* $\big|p^{-1}(\{x\})\big| = \big[\pi_1(X, x) : p_*\big(\pi_1(\tilde{X}, \tilde{x})\big)\big]$.
- *(c) p is a homeomorphism $\iff$ $p_*\colon \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is surjective.*

4 $\pi_1(S^1) \cong \mathbb{Z}$ and its consequences

November 9, 2023

Lemma 4.1. *Let $p\colon \mathbb{R} \to S^1$ be given by $t \mapsto e^{i2\pi t}$. Let $\alpha \in \operatorname{Path}(S^1; x, y)$ and $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(\{x\})$. Then*

$$\alpha^{\sim}_{\tilde{x}_2} = T_{\tilde{x}_2 - \tilde{x}_1} \circ \alpha^{\sim}_{\tilde{x}_1}$$

where $T_{\tilde{x}_2 - \tilde{x}_1}$ denotes translation by $\tilde{x}_2 - \tilde{x}_1$.

Corollary 4.2 ($\pi_1(S^1) \cong \mathbb{Z}$)**.** *Let $p\colon \mathbb{R} \to S^1$ be given by $t \mapsto e^{i2\pi t}$. Then the following defines a group isomorphism $\pi_1(S^1, 1) \to \mathbb{Z}$:*

$$\begin{aligned} [\alpha] &\mapsto [\alpha]\,0 \\ &= \alpha^{\sim}_0(1) \end{aligned}$$

Corollary 4.3. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.*

Proposition 4.4. *Borsuk-Ulam (Proposition 7.1) holds for $n = 2$.*

Proposition 4.5 (Brower for $n = 2$)**.** *Any continuous function $D^2 \to D^2$ has a fixed point.*

Remark. *Brower for $n = 1$ follows from the intermediate value theorem.*

Appendix A

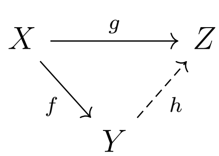
Things used

1 Set theoretic facts

September 19, 2023

Proposition 1.1 (Surjections are epic in Set)**.** *Let $f\colon X \to Y$ and $g\colon X \to Z$ be set theoretic functions with f surjective. Let $\sim_f$, $\sim_g$ be the equivalence relations induced on X due to f, g. Then the following hold:*

(i) There exists at most one $h\colon Y \to Z$ making the following diagram commute:



(ii) The following are equivalent:

- *(a) There exists $h\colon Y \to Z$ making the diagram commute.*
- *(b) Each $[x]_{\sim_f} \subseteq [x]_{\sim_g}$.*

(iii) Let $h\colon Y \to Z$ such that the diagram commutes. Then the following hold:

- *(a) $\operatorname{range} h = \operatorname{range} g$.*
- *(b) h is injective $\iff$ $\sim_f$ and $\sim_g$ coincide.*

Lemma 1.2. *Let $f\colon X \to Y$ be a set theoretic function, $A \subseteq X$ and $b \in Y$. Then*

$$b \in f(A) \iff f^{-1}(\{b\}) \cap A \neq \emptyset.$$

2 Topological things

September 20, 2023

2.1 Disjoint union topology

September 20, 2023

Proposition 2.1 (Disjoint union topology)**.** *Given spaces X_i's, the following defines a topology on $X := \dot\bigcup_\alpha \tilde{X}_\alpha$ where $\tilde{X}_\alpha := X_\alpha \times \{\alpha\}$:*[1]

$$\{U \subseteq X : \text{each } U \cap \tilde{X}_\alpha \text{ is open in } \tilde{X}_\alpha\}$$

Notation. *This space is denoted by $\bigsqcup_\alpha X_\alpha$.*

Proposition 2.2 (Disjoint union topology generalizes the subspace topology on disjoint open sets)**.** *Let U_α be disjoint open subsets of X and $V \subseteq \dot\bigcup_\alpha U_\alpha$. Then the following hold:*

(i) V is open in $\dot\bigcup_\alpha U_\alpha \iff$ each $V \cap U_\alpha$ is open in U_α.

(ii) $\dot\bigcup_\alpha U_\alpha \cong \bigsqcup_\alpha U_\alpha$.

Proposition 2.3. *If each $X_\alpha \cong Y_\alpha$, then $\bigsqcup_\alpha X_\alpha \cong \bigsqcup_\alpha Y_\alpha$.*

Corollary 2.4. *Let X be a space and Λ be an indexing set, considered under the discrete topology. Then $\bigsqcup_{\alpha \in \Lambda} X$ and $X \times \Lambda$ have the same topology.*[2]

2.2 (Weak) local path-connectedness

November 12, 2023

Definition 2.5 ((Weakly) locally path connected)**.** A space X with a point $p \in X$ is called

(i) *weakly locally path connected at p* iff each open neighborhood of p contains a path connected neighborhood of p; and,

(ii) *locally path connected at p* iff each open neighborhood of p contains a path connected open neighborhood of p.

X is called *(weakly) locally path connected* iff it is so at each point in it.

[1] Note that there is only one topology possible on $\{\alpha\}$, so that the (product) topologies $\tilde{X}_\alpha$'s are uniquely determined.

[2] Note that they are equal as sets in the first place.

2.3 Local homeomorphisms

November 12, 2023

Definition 2.6 (Local homeomorphisms)**.** A continuous $f\colon X \to Y$ is called a local homeomorphism iff each point in X has an open neighborhood on which f's restriction is a homeomorphism.

Corollary 2.7. *Restrictions of local homeomorphisms are local homeomorphisms.*

3 Categorical ideas

November 12, 2023

Definition 3.1 (Lifts)**.** In a category, let $p\colon E \to B$ and $f\colon X \to B$ be morphisms. Then a p-lift of f is any morphism $\tilde{f}\colon X \to E$ making the following diagram commute: