Contents

Chapter I

Basic stuff

1 Categories

September 16, 2023

Definition 1.1 (Category)**.** A category C consists of a class of objects $\operatorname{Obj}\mathsf{C}$, and for every pair of objects X, Y of C, a set of morphisms $\operatorname{Hom}_{\mathsf{C}}(X,Y)$ such that the following hold:

(i) *(Composition of morphisms)***.** For any objects X, Y, Z of C, there's a function:

$$\operatorname{Hom}_{\mathsf{C}}(X,Y) \times \operatorname{Hom}_{\mathsf{C}}(Y,Z) \to \operatorname{Hom}_{\mathsf{C}}(X,Z)$$
$$(f,g) \mapsto gf$$

(ii) *(Associativity of composition)***.** For $f \in \operatorname{Hom}_{\mathsf{C}}(X,Y)$, $g \in \operatorname{Hom}_{\mathsf{C}}(Y,Z)$, $h \in \operatorname{Hom}_{\mathsf{C}}(Z,W)$, we have

$$h(gf) = (hg)f.$$

(iii) *(Identity morphisms)***.** For any object X of C, we have $1_X \in \operatorname{Hom}_{\mathsf{C}}(X,X)$[1] such that for any $f \in \operatorname{Hom}_{\mathsf{C}}(X,Y)$ and for any $g \in \operatorname{Hom}_{\mathsf{C}}(Y,X)$, we have

$$f\,1_X = f, \text{ and}$$
$$1_X\,g = g.$$

(iv) *(Domains and codomains of morphisms)***.** For any objects X, Y, Z, W of C, the sets $\operatorname{Hom}_{\mathsf{C}}(X,Y)$ and $\operatorname{Hom}_{\mathsf{C}}(Z,W)$ are disjoint unless $X = Z$ and $Y = W$.

If $\operatorname{Obj}(\mathsf{C})$ forms a set, then C is said to be *small*.

[1] It's easily seen that such a morphism is unique for each X, justifying the notation 1_X.

Notation. *For a category* C*, we'll also write "*$f\colon X \to Y$ *in* C*" to mean that* X, Y *are objects in* C *and* $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$*.*

Remark. *What we have defined are actually* locally small *categories. In general the hom-sets needn't be small. However, generally, a small category is one whose objects as well as morphisms form a set.*

Definition 1.2 (Subcategories)**.** A category D is said to be a subcategory of a category C iff the following hold:

(i) $\mathrm{Obj}(\mathsf{D})$ is a subclass of $\mathrm{Obj}(\mathsf{C})$.

(ii) $\mathrm{Hom}_{\mathsf{D}}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all objects A, B of C.

(iii) The morphism composition in D is inherited from that in C.

If the "$\subseteq$" in (ii) is "$=$", then D is called a *full* subcategory of C.

Example 1.3 (Some examples)**.**

(i) *(Subcategories of* Set*)*. Set, Top, Grp, Rng, Ring, Vect_K, Mod_R, their pointed versions.

(ii) *(A category whose objects needn't be sets)*. Let R be a relation on a set X which is reflexive and transitive. Then the elements of X form a category with the set of morphisms from a to b being

$$\begin{cases} \{(a,b)\}, & a \mathrel{R} b \\ \emptyset, & \text{otherwise} \end{cases}$$

and the morphisms being given by

$$(b,c)(a,b) = (a,c).$$

(iii) *(A category whose objects are sets but morphisms are not set theoretic functions)*. RelHTop.

(iv) *(Opposite category)*. Let C be a category. Then the objects of C form another category C^{op} with $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(Y, X)$ and composition of $f\colon X \to Y$ and $g\colon Y \to Z$ in C^{op} given by the composition of g and f in C.

2 Monics, epics and isomorphisms

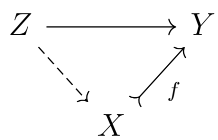
September 17, 2023

Definition 2.1 (Monomorphisms and epimorphisms)**.** Let $f\colon X \to Y$ be a morphism in a category. Then:

(i) f is called a monomorphism (or monic) iff

$$fu = fv \implies u = v$$

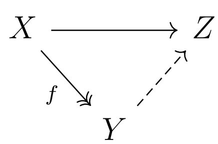
for any morphisms $u, v\colon Z \to X$, or equivalently, any morphism $Z \to Y$ factors through f via at most one morphism:



(ii) f is called an epimorphism (or epic) iff

$$uf = vf \implies u = v$$

for any morphisms $u, v\colon Y \to Z$, or equivalently, any morphism $X \to Z$ factors through f via at most one morphism:



Notation. *We'll sometimes denote monics by* $\rightarrowtail$ *and epics by* $\twoheadrightarrow$ *.*

Definition 2.2 (Inverses and isomorphisms)**.** A morphism $f\colon X \to Y$ is called an isomorphism iff there exists another morphism $g\colon Y \to Z$ such that

$$gf = 1_X, \text{ and}$$
$$fg = 1_Y.$$

Such a morphism is called an inverse of f, and is denoted by f^{-1}.[2]

[2] Uniqueness of inverses (an easy fact) justifies this notation.

Corollary 2.3.

(i) *Inverse of an isomorphism is an isomorphism.*

(ii) *An isomorphism is monic and epic.*

(iii) *In subcategory* C *of* Set*, injective morphisms are monic and sujective morphisms are epic. Further, if* C *contains all the identity functions of its objects, then isomorphisms are bijective.*

(iv) *Composition of monics (respectively epics) is monic (respectively epic).*

(v) *(a)* gf *is monic* $\implies$ f *is monic.*
(b) gf *is epic* $\implies$ g *is epic.*

Example 2.4 (Counters to the converse of (iii))**.**

(i) *(Monic* $\not\Rightarrow$ *injective)***.** In the category of "root-able" groups[3], $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic.

(ii) *(Epic* $\not\Rightarrow$ *surjective)***.**

(a) In the category of Hausdorff spaces, any inclusion $E \hookrightarrow X$ with E dense in X, is epic.[4]

(b) In the category of commutative rings with identities with homomorphisms preserving identities, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic.[5]

Proposition 2.5. *Monics are injective and epics, surjective in* Grp*.*

3 Initials, terminals, and zeroes

September 24, 2023

Definition 3.1 (Initial, terminals, and zeroes)**.** An object X in a category C is called

(i) *initial* iff $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ is a singleton for each object Y;

(ii) *terminal* iff $\mathrm{Hom}_{\mathsf{C}}(Y, X)$ is a singleton for each object Y; and,

(iii) *zero* iff it is initial as well as terminal.

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[3] A group G is "root-able" iff $G = G^n$ for every $n \in \mathbb{Z} \setminus \{0\}$. If G is abelian as well, we call it divisible, and in the additive notation, the condition reads $G = nG$. Fact: Any finite "root-able" group is trivial.

[4] More general: A continuous function on a Hausdorff codomain is determined by its restriction on a dense subset of its domain.

[5] More strongly, any homomorphism $I \to R$ from an integral domain I extends uniquely to $\mathrm{Frac}(I) \to R$.

Corollary 3.2.

(i) Initials (respectively terminals, zeroes) are unique up to unique isomorphisms.

(ii) Any morphism into an initial object is epic and any morphism from a terminal object is monic.

4 Zero morphisms

September 24, 2023

Definition 4.1 (Left and right zeroes, and zero morphisms)**.** A morphism f in a category is called a

(i) *left zero* iff whenever defined, $fu = fv$ for any morphisms u, v;

(ii) *right zero* iff whenever defined, $uf = vf$ for any morphisms u, v; and,

(iii) *zero morphism* iff it's both, a left as well as a right zero.

Remark. *"a followed by b" will mean ba and not ab.*

Corollary 4.2.

(i) The morphisms associated with initial (terminal) objects are right (left) zeroes.

(ii) If f is a left zero, then fu, whenever defined, is too. Similarly for right zeroes.

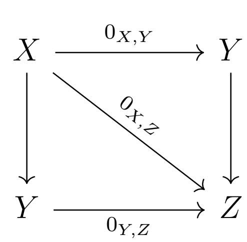
(iii) A right zero followed by a left zero is a zero morphism.

(iv) If z is a zero morphism, then vzu, whenever defined is a zero morphism.

(v) If 0 is a zero object, then $X \to 0 \to Y$ is a zero morphism.

(vi) In a category, there exists at most one family of zero morphisms between each pair of objects.

Definition 4.3 (A category having compatible zero morphisms)**.** A category is said to have compatible zero morphisms iff there exists a family of zero morphisms $0_{X,Y} \colon X \to Y$ for each pair of objects such that for any morphisms $X \to Y$ and $Y \to Z$, the following diagram commutes:



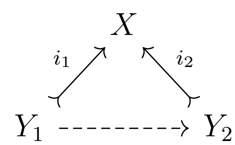
Corollary 4.4. *If a category has a zero object* 0*, then the morphisms* $X \to 0 \to Y$ *form a compatible family of zero morphisms.*

5 Subobjects and quotients

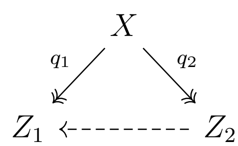
September 24, 2023

Definition 5.1 (Subobjects, quotients and their comparisons)**.** In a category, *subobjects* of an object X are monics with codomain X, and *quotients* of X are epics with domain X.

If $i_1 \colon Y_1 \to X$ and $i_2 \colon Y_2 \to X$ are subobjects of X, then we write $i_1 \leq i_2$ iff i_2 factors through i_1:



Similarly, for quotients $q_1 \colon X \to Z_1$ and $q \colon X \to Z_2$ of X, we write $q_1 \leq q_2$ iff q_2 factors through :

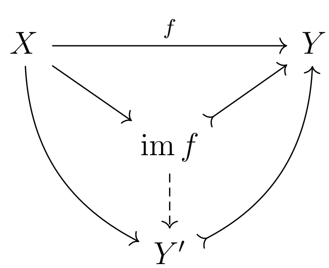


Corollary 5.2.

(i) If $f \colon Y \to X$ *is a subobject of* X *and* $g \colon Z \to Y$ *a subobject of* Y*, then* gf *is a subobject of* X*. Similarly for quotients.*

(ii) The comparisons defined in Definition 5.1 form partial order with the "equality" replaced with "being isomorphic".

Definition 5.3 (Images)**.** An image of a morphism $f \colon X \to Y$ is a smallest subobject of Y through which f factors:



Notation. *The arrow* im $f \to Y$ *stands for any image of* f*, which are all isomorphic, due to Corollary 5.4.*

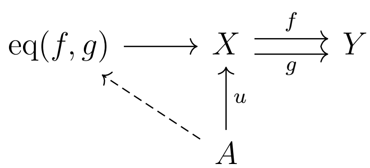
Corollary 5.4. *Images are unique up to unique isomorphisms.*

6 (Co)equalizers and (co)products

October 25, 2023

Remark. *Whenever the domain and codomain are clear from the context, we'll omit the subscript from* $0_{X,Y}$.

Definition 6.1 ((Co)equalizers and (co)kernels)**.** An equalizer of $f, g\colon X \to Y$ in a category is a morphism which is terminal among all the morphisms $u\colon A \to X$ such that $fu = gu$:



If the category has zero morphisms, then an equalizer of f and $0_{X,Y}$ is called a *kernel of* f.

Coequalizers and *cokernels* are defined dually.

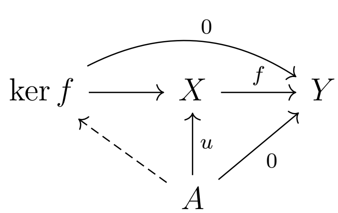
Notation. *The arrow* $\mathrm{eq}(f, g) \to X$ *stands for any equalizer of* f*,* g*, which are all isomorphic due to Corollary 6.2 (ii). Similarly, we use* $\ker(f) \to X$*,* $Y \to \mathrm{coeq}(f, g)$*, and* $Y \to \mathrm{coker}(f)$*.*

Corollary 6.2.

(i) Equalizers are monic and coequalizers, epic.

(ii) (Co)equalizers are unique up to unique isomorphisms.

(iii) (Characterizing (co)kernels)**.** *In a category with zero morphisms, a kernel* $\ker f \to X$ *of* $f\colon X \to Y$ *is characterized by being the terminal among all the morphisms* $u\colon A \to X$ *such that* $fu = 0$*:*



Dual characterization holds for cokernels.

(iv) *Taking kernels of quotients, or cokernels of subobjects, reverses order.*

Proposition 6.3 (Quotient topology as a coequalizer in Top)**.** *Let X be a topological space and $\sim$ an equivalence relation on X and $R \subseteq X \times X$ be the corresponding subset. Consider $X/{\sim}$ under quotient topology and R under the subspace topology. Then $p\colon X \to X/{\sim}$ is a coequalizer of the projections $R \to X$.*

7 (Co)products

December 10, 2023

Definition 7.1 ((Co)products)**.** A product of objects X_i's in a category is an object P together with morphisms $\pi_i\colon P \to X_i$'s which is terminal among all families of morphisms $f_i\colon Y \to X_i$:

$$\begin{array}{ccc} & X_i & \\ {}^{f_i}\nearrow & & \nwarrow^{\pi_i} \\ Y & \dashrightarrow & P \end{array}$$

Coproducts are defined dually.

Corollary 7.2. *(Co)products are unique up to unique isomorphisms.*

8 Abelian categories

October 25, 2023

Definition 8.1 ((Pre-)additive and (pre-)abelian categories)**.** A category C is called:

(i) *pre-additive* iff the following hold:
   (a) C has a zero object 0.
   (b) Each $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ forms an additive abelian group with 0 being the additive identity.
   (c) Composition of morphisms is bilinear.

(ii) *additive* iff it is pre-additive and has finite products and coproducts.

(iii) *pre-abelian* iff it is additive and has kernels and cokernels.

(iv) *abelian* iff it is pre-abelian, and all monics arise as kernels and epics as cokernels.

Proposition 8.2. *In a pre-additive category, the following hold:*

(i) *Finite products and coproducts coincide.*[6]

(ii) 1_X *is a kernel, and* 1_Y *a cokernel of* $0_{X,Y}$.

(iii) *For* $f\colon X \to Y$,

  (a) $0 \to X$ *is a kernel of* $f \iff f$ *is monic.*

  (b) $Y \to 0$ *is a cokernel of* $f \iff f$ *is epic.*

(iv) *A kernel of a cokernel of a kernel is a kernel, and a cokernel of a kernel of a cokernel is a cokernel.*

Corollary 8.3. *In an abelian category, a monic is a kernel of any cokernel of itself, while an epic is a cokernel of any kernel of itself.*

Proposition 8.4. *In an abelian category, a morphism is an isomorphism* $\iff$ *it is monic and epic.*

Definition 8.5 (Exact sequences)**.** In a pre-additive category, a sequence (finite or infinite at either end) of morphisms

$$\cdots \xrightarrow{f_{i-1}} X_{i-1} \xrightarrow{f_i} X_i \xrightarrow{f_{i+1}} X_{i+1} \longrightarrow \cdots$$

is said to be exact at a nonterminal object X_i iff kernels of f_{i+1} are precisely the images of f_i.

The sequence is called exact iff it is exact at all its nonterminal objects.

Proposition 8.6 (Characterizing exactness)**.** *In an abelian category, a sequence of morphisms*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact $\iff$ *the following compositions are* 0*:*

$$A \xrightarrow{f} B \xrightarrow{g} C$$
$$\ker g \longrightarrow B \longrightarrow \operatorname{coker} f$$

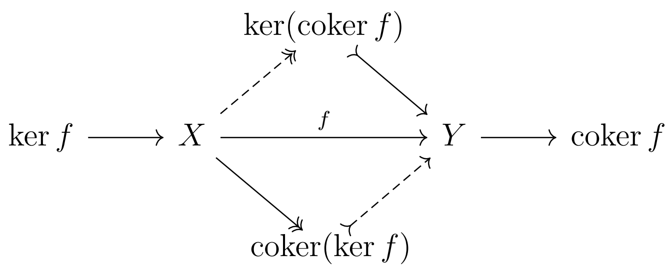
Corollary 8.7 (Using exactness to characterize various things)**.** *In an abelian category, the exactness of the sequence of the left is equivalent to the property stated at*

[6] Only at the object level.

the right:

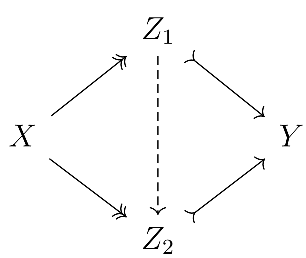
$0 \longrightarrow X \xrightarrow{f} Y$	*f is monic*
$X \xrightarrow{f} Y \longrightarrow 0$	*f is epic*
$0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$	*f is an isomorphism*
$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$	*f is a kernel of g*
$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$	*g is a cokernel of f*
$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$	*f is a kernel of g and g a cokernel of f*

Theorem 8.8. *In an abelian category, any morphism $f\colon X \to Y$ admits unique*[7] *factorizations as follows:*



Proposition 8.9 ($\operatorname{im} f = \ker(\operatorname{coker} f)$)**.** *In an abelian category, a kernel of any cokernel of f is an image of f.*

Corollary 8.10 (Unique factorization)**.** *In an abelian category, the commutativity of solid arrows below implies the existence of the unique dashed arrow which makes the resulting diagram commute:*



Further, the dashed arrow is an isomorphism.

[7] Once $\ker f$, $\operatorname{coker} f$, $\operatorname{coker}(\ker f)$ and $\ker(\operatorname{coker} f)$ have been fixed.

Proposition 8.11 (Characterizing abelian-ness)**.** *Let* C *be a pre-abelian category. Then the following hold:*

(i) For every $f\colon X \to Y$*, there exists a unique*[8] $\tilde{f}$ *such that the following diagram commutes:*[9]

$$\begin{array}{ccccc} \ker f & \longrightarrow & X & \longrightarrow & \operatorname{coker}(\ker f) \\ & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \tilde{f}} \\ \operatorname{coker} f & \longleftarrow & Y & \longleftarrow & \ker(\operatorname{coker} f) \end{array}$$

(ii) C *is abelian* $\iff$ $\tilde{f}$ *is an isomorphism for every* f*.*

9 Functors

December 10, 2023

Definition 9.1 (Functors)**.** A functor $F\colon \mathsf{C} \to \mathsf{D}$ is an assignment of objects and morphisms of C to those in D such that the following hold:

(i) F is compatible with domains and codomains:

$$\begin{array}{ccc} A & & F(A) \\ {\scriptstyle f}\big\downarrow & \overset{F}{\longmapsto} & \big\downarrow{\scriptstyle F(f)} \\ B & & F(B) \end{array}$$

(ii) F is compatible with composition, *i.e.*, commutativity of the diagram on the left implies that on the right:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & B & \end{array} \qquad \overset{F}{\longmapsto} \qquad \begin{array}{ccc} F(A) & \xrightarrow{F(h)} & F(C) \\ & {\scriptstyle F(f)}\searrow \quad \nearrow{\scriptstyle F(g)} & \\ & F(B) & \end{array}$$

(iii) F is compatible with identities:

$$\begin{array}{ccc} A & & F(A) \\ {\scriptstyle 1_A}\big\downarrow & \overset{F}{\longmapsto} & \big\downarrow{\scriptstyle 1_{F(A)}} \\ A & & F(A) \end{array}$$

[8]Once $\ker f$, $\operatorname{coker} f$, $\operatorname{coker}(\ker f)$ and $\ker(\operatorname{coker} f)$ have been fixed.

[9]$\operatorname{coker}(\ker f)$ is shortened for $\operatorname{coker}(\ker f \to X)$. Similarly for $\ker(\operatorname{coker} f)$.

Example 9.2 (Hom-functors)**.** Let A, B be objects of a category C. Then the following defines the functor $\mathrm{Hom}_{\mathsf{C}}(A, -)\colon \mathsf{C} \to \mathsf{Set}$:

$$\begin{array}{ccc} X & & \mathrm{Hom}_{\mathsf{C}}(A,X) \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathsf{C}}(A,f)} \\ Y & & \mathrm{Hom}_{\mathsf{C}}(A,Y) \end{array}$$

where $\mathrm{Hom}_{\mathsf{C}}(A, f)$ is given by $\alpha \mapsto f\alpha$.

We also have the functor $\mathrm{Hom}_{\mathsf{C}}(-, B)\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ given by:[10]

$$\begin{array}{ccc} X & & \mathrm{Hom}_{\mathsf{C}}(X,B) \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\uparrow{\scriptstyle \mathrm{Hom}_{\mathsf{C}}(f,B)} \\ Y & & \mathrm{Hom}_{\mathsf{C}}(Y,B) \end{array}$$

where $\mathrm{Hom}_{\mathsf{C}}(g, B)$ is given by $\beta \mapsto \beta f$.[11]

Corollary 9.3.

(i) *Functors preserve isomorphisms.*

(ii) (Composition of functors)**.** *Let* $F\colon \mathsf{C} \to \mathsf{D}$ *and* $G\colon \mathsf{D} \to \mathsf{E}$ *be functors. Then the following defines their composite functor* $GF\colon \mathsf{C} \to \mathsf{E}$*:*

$$\begin{array}{ccc} A & & G(F(A)) \\ {\scriptstyle f}\big\downarrow & \overset{GF}{\longmapsto} & \big\downarrow{\scriptstyle G(F(f))} \\ B & & G(F(B)) \end{array}$$

(iii) (Identity functors)**.** *For any category* C*, we have an identity functor* 1_{C}*:*

$$\begin{array}{ccc} A & & A \\ {\scriptstyle f}\big\downarrow & \overset{1_{\mathsf{C}}}{\longmapsto} & \big\downarrow{\scriptstyle f} \\ B & & B \end{array}$$

(iv) (Category of categories)**.** *Small categories with functors as morphisms form a category* Cat.

[10] Note that $f\colon X \to Y$ is in C.
[11] The composition is taking in C.

Definition 9.4 (Isomorphism of categories)**.** A functor $F\colon \mathsf{C} \to \mathsf{D}$ is called an isomorphism iff there exists another functor $G\colon \mathsf{D} \to \mathsf{C}$ such that $GF = 1_{\mathsf{C}}$ and $FG = 1_{\mathsf{D}}$.[12]

Definition 9.5 (Full, faithful and essentially surjective functors)**.** A functor $F\colon \mathsf{C} \to \mathsf{D}$ is called *full* (respectively *faithful*) iff for all objects A, B of C, the associated function $\mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{D}}(F(A), F(B))$ is surjective (respectively injective).

F is called *essentially surjective* iff for every object X of D, there exists an object A of C such that $F(A)$ is isomorphic to X.

Lemma 9.6 (Fully faithful functors are conservative)**.** *Let $F\colon \mathsf{C} \to \mathsf{D}$ be fully faithful and f be a morphism in C. Then f is an isomorphism $\iff$ $F(f)$ is an isomorphism.*

10 Natural transformations

December 11, 2023

Definition 10.1 (Natural transformations)**.** Let $F, G\colon \mathsf{C} \to \mathsf{D}$ be functors. Then a natural transformation $\eta\colon F \Rightarrow G$ is a family of morphisms η_A in D indexed by the objects A on C such that the following hold:

(i) $\eta_A\colon F(A) \to G(A)$ in D.

(ii) For any morphism $f\colon A \to B$ in C, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\eta_B]{} & G(B) \end{array}$$

If each η_A is an isomorphism, then η is called a natural *isomorphism*.

Notation. *We'll denote $\mathrm{Nat}(F, G)$ to denote the class of all natural transformations $F \Rightarrow G$.*

Corollary 10.2.

[12] Clearly, for small categories, isomorphism of categories is precisely an isomoirphism between them in Cat.

(i) (Composition of natural transformations). *Let* $F, G, H\colon \mathsf{C} \to \mathsf{D}$ *be functors and* $\eta\colon F \Rightarrow G$ *and* $\xi\colon G \Rightarrow H$ *be natural transformations. Then the following defines the composite natural transformation* $\xi\eta\colon F \Rightarrow H$*:*

$$(\xi\eta)_A := \xi_A\,\eta_A$$

(ii) (Identity natural transformations). *Let* $F\colon \mathsf{C} \to \mathsf{D}$ *be a functor. Then the following defines the identity natural transformation* $1_F\colon F \Rightarrow F$*:*

$$(1_F)_A := 1_{F(A)}$$

(iii) (Functor category). *Let* C *be a small category. Then for any category* D*, the functors* $\mathsf{C} \to \mathsf{D}$ *form a category* $\mathrm{Funct}(\mathsf{C}, \mathsf{D})$ *with the morphisms being the natural transformations between them.*
Further, a natural isomorphism between functors $\mathsf{C} \to \mathsf{D}$ *is precisely an isomorphism in* $\mathrm{Funct}(\mathsf{C}, \mathsf{D})$*.*

Definition 10.3 (Equivalence of categories)**.** A functor $F\colon \mathsf{C} \to \mathsf{D}$ is called an equivalence iff there exists another functor $G\colon \mathsf{D} \to \mathsf{C}$ such that GF is naturally isomorphic to 1_{C} and FG to 1_{D}.

Theorem 10.4 (Characterizing equivalence)**.** *Let* $F\colon \mathsf{C} \to \mathsf{D}$ *be a functor. Then* F *is an equivalence* $\iff$ F *is fully faithful and essentially surjective.*[13]

11 Yoneda lemma

December 11, 2023

Proposition 11.1 (Yoneda lemma)**.** *Let* C *be a small category.*[14] *Let* $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ *be a functor and* $B \in \mathrm{Obj}(\mathsf{C})$*. Then the function* $\mathrm{Nat}\big(\mathrm{Hom}_{\mathsf{C}}(-, B), F\big) \to F(B)$ *given by*

$$\eta \mapsto \eta_B(1_B)$$

is a bijection with the inverse given by

$$c \mapsto \left\{\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(X, B) &\to F(X) \\ f &\mapsto F(f)(c) \end{aligned}\right\}_{X \in \mathrm{Obj}(\mathsf{C})}$$

[13] "$\Leftarrow$" requires choice for classes.
[14] For $\mathrm{Nat}\big(\mathrm{Hom}_{\mathsf{C}}(-, Y), F\big)$ to be a set.

Lemma 11.2 (The canonical functor $\mathsf{C} \to \mathrm{Funct}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$)**.** *Let* C *be a small category.*[15] *Then the following defines a functor* $\mathsf{C} \to \mathrm{Funct}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$*:*

$$\begin{array}{ccc} A & & \mathrm{Hom}_{\mathsf{C}}(-, A) \\ {\scriptstyle f}\downarrow & \longmapsto & \Downarrow{\scriptstyle \eta(f)} \\ B & & \mathrm{Hom}_{\mathsf{C}}(-, B) \end{array}$$

where $\eta(f)$ *is the natural transformation given by*

$$\begin{aligned} \eta(f)_X \colon \mathrm{Hom}_{\mathsf{C}}(X, A) &\longrightarrow \mathrm{Hom}_{\mathsf{C}}(X, B) \\ \alpha &\longmapsto f\alpha. \end{aligned}$$

Proposition 11.3 (Yoneda theorem)**.** *Let* C *be a small category. Then the canonical functor* $\mathsf{C} \to \mathrm{Funct}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ *is fully faithful.*

[15] So that we can talk about $\mathrm{Funct}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$.