Organized results
Point-Set Topology
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Part I

Normed linear spaces and metric spaces

Chapter 1

Normed linear spaces

February 1, 2022

Remark 1.0.1. We'll let $\mathbb{K}$ stand for either $\mathbb{R}$ or $\mathbb{C}$ (viewed as fields). We'll use $\mathbb{F}$ for a general field.

Definition 1.0.2 (Normed linear spaces)**.** Let V be a vector space over $\mathbb{K}$. A norm on V is a function $\|\cdot\|: V \to [0, \infty)$ such that for any $x, y \in V$ and any scalar α, we have

(a) $\|x\| = 0 \iff x = 0$,
(b) $\|\alpha x\| = |\alpha| \|x\|$, and
(c) $\|x + y\| \leq \|x\| + \|y\|$.

Remark 1.0.3. *Prove Minkowski inequality!*

Proposition 1.0.4 (Norms on $\mathbb{K}^n$)**.** *Let $n \geq 1$ and $p \in [1, \infty)$. Then the following are norms on $\mathbb{K}^n$ over $\mathbb{K}$ with the usual vector addition and scalar multiplication.*

(a) (l_p-norms)**.**

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

(b) (max norm)**.**

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |x_i|$$

Remark 1.0.5. For $n = 1$, all the above norms coincide with the absolute value norm. Hence, no need to discuss that.

Proposition 1.0.6 (Norms on infinite dimensional vector spaces). *Let $p \in [1, \infty)$. In all of the following cases, the defined set will be a vector space over $\mathbb{K}$ under the usual operations of addition and scalar multiplication for sequences in $\mathbb{K}$, and the defined function will be a norm over it.*

(a) (For direct sums of copies of $\mathbb{K}$). *Let I be a nonempty set and*

$$V := \left\{ x \in \mathbb{K}^I : x_i \neq 0 \textit{ for only finitely many } i\textit{'s in } I \right\}.$$

Define $\|\cdot\|_p, \|\cdot\|_\infty : V \to [0, \infty)$ as

$$\|x\|_p := \left(\sum_{i \in I} |x_i|^p \right)^{1/p},$$
$$\|x\|_\infty := \sup_{i \in I} \|x_i\|.$$

(b) (For l^p spaces). *Let*

$$l^p := \left\{ x \in \mathbb{K}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

and define $\|\cdot\|_p : l^p \to [0, \infty)$ as

$$\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

(c) (For l^∞ space). *Let*

$$l^\infty := \left\{ x \in \mathbb{K}^{\mathbb{N}} : \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}$$

and define $\|x\|_\infty : l^\infty \to [0, 1)$ as

$$\|x\|_\infty := \sup_{i \in \mathbb{N}} |x_i|.$$

Proposition 1.0.7 (Subspaces of normed linear spaces). *Let V be a normed linear space and W be a subspace of V. Then W is also a normed linear space with the inherited norm.*

Remark 1.0.8. The restriction of the norms in Proposition 1.0.6 to the copies of $\mathbb{K}^n$ inside them agrees with the corresponding norms in Proposition 1.0.4. Hence, we'll call the l_∞-norm to mean the appropriate norm when talking of $\mathbb{K}^n$ or $\mathbb{K}^I$ or l^∞. Similarly, we'll call l_p norm for $\mathbb{K}^n$ or $\mathbb{K}^I$ or l^p.

Proposition 1.0.9 (Inclusions of l^p spaces)**.** *Let $1 \leq p < q \leq \infty$. Then $l^p \subsetneq l^q$.*

Remark 1.0.10. For proper inclusion, consider the sequence $(1/n^s)_{n=1}^\infty$ for any appropriate value of s.

Remark 1.0.11. For any $n \geq 1$, the unit discs for l_p-norm in $\mathbb{K}^n$ approach the unit disc for the l_∞-norm as $p \to \infty$. *Does this hold for l^p spaces as well?*

Proposition 1.0.12 (Inclusion of l^p discs)**.** *Let $n \geq 2$ and $1 \leq p < q \leq \infty$. Let D_p and D_q stand for the corresponding unit discs for the respective norms in $\mathbb{K}^n$. Then $D_p \subsetneq D_q \iff p < q$.*

Remark 1.0.13. For proper inclusions, consider $(a, \cdots, a)$ for an appropriate $a \in \mathbb{K}$.

Corollary 1.0.14. *Let $n \geq 1$ and $1 \leq p, q \leq \infty$. Then the following are equivalent:*

(a) $p < q$.
(b) $\|x\|_q \leq \|x\|_p$ *for all* $x \in \mathbb{K}^n$.

Proposition 1.0.15 (Inequality between l_p-norms)**.** *Let $n \geq 1$ and $x \in \mathbb{K}^n$. Let $p \in [1, \infty)$. Then*

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty.$$

1.1 Commutative algebras over $\mathbb{K}$

February 2, 2022

Definition 1.1.1 (Commutative algebras over $\mathbb{F}$)**.** Let $\mathbb{F}$ be a field and A be a set equipped with operations of addition and multiplication from $\mathbb{A} \times A$ to A such that $(x, y) \mapsto x + y, xy$ and a scalar multiplication from $\mathbb{F} \times A$ to A such that $(\alpha, x) \mapsto \alpha x$. Then A is a commutative algebra over $\mathbb{F}$ iff

(a) A is a vector space over $\mathbb{K}$ with addition and scalar multiplication,
(b) A is a commutative ring with unity with addition and multiplication, and
(c) $\alpha(xy) = (\alpha x)y$ for all $\alpha \in \mathbb{F}$ and all $x, y \in A$.

Proposition 1.1.2 (Commutative algebra of functions into a field)**.** *Let X be a set and $\mathbb{F}$ a field. Then $\mathbb{K}^X$ is a commutative algebra over $\mathbb{F}$ with the usual pointwise addition and multiplication of functions, and the usual scalar multiplication.*

Definition 1.1.3 (Subalgebras)**.** Let A be a commutative algebra over a field $\mathbb{F}$, and $S \subseteq A$. Then S is a subalgebra of A iff it is closed under the operations algebra operations on A, and it forms a commutative algebra over $\mathbb{F}$ with the inherited operations.

Lemma 1.1.4. *Let A be a commutative algebra over a field $\mathbb{F}$ and $S \subseteq A$. Then S is a subalgebra of A $\iff$ it is closed under the algebra operations on A, and if S contains the identity of multiplication on A.*

Proposition 1.1.5 (Polynomials as subalgebra)**.** *Let $\mathbb{F}$ be a field and $n \geq 1$. Then under the usual pointwise operations, $\mathbb{F}[x_1, \ldots, x_n]$ is a subalgebra of $\mathbb{K}^{(\mathbb{K}^n)}$.*

Definition 1.1.6 (Normed algebras)**.** Let A be a commutative algebra over $\mathbb{K}$ and $\|\cdot\|$ be a norm on A as a vector space over $\mathbb{K}$. Then $\|\cdot\|$ is a norm on the A as an algebra iff $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

Proposition 1.1.7 ($B(X, \mathbb{K})$ as a normed algebra)**.** *Let X be a set and let $B(X, \mathbb{K})$ be the set of bounded functions from X to $\mathbb{K}$. Then $(B, \mathbb{K})$ is a subalgebra of $\mathbb{K}^X$ over $\mathbb{K}$.*

Define $\|\cdot\|_\infty \colon B(X, \mathbb{K}) \times B(X, \mathbb{K}) \to [0, \infty)$ as

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$

This makes $B(X, \mathbb{K})$ a normed algebra.

Proposition 1.1.8 (Convergence in $\|\cdot\|_\infty$)**.** *Let X be a set and consider $B(X, \mathbb{K})$ as the usual normed algebra. Then the convergence (with respect to the induced metric) of a sequence of functions in $B(X, \mathbb{K})$ is equivalent to their usual uniform convergence.*[1]

[1] For uniform convergence, we don't need X to be a metric space.

Proposition 1.1.9 (*Prove this!*). *Let X be a set. Consider $B(X, \mathbb{K})$ as the usual normed algebra. Then it is complete.*[2]

Further, if X is a metric space,[3] *then the subset $C(X, \mathbb{K})$ of $B(X, \mathbb{K})$ containing all the continuous functions in $B(X, \mathbb{K})$, is a normed subalgebra (with the inherited norm), and it is complete.*

[2]Every Cauchy sequence (with respect to the induced metric) is convergent.
[3]This is needed to talk of continuous functions.

Chapter 2

Metric spaces

February 2, 2022

Definition 2.0.1 (Metric spaces)**.** Let X be a set. Then a metric on X is a function $d\colon X \times X \to [0, \infty)$ such that for any $x, y, z \in X$, we have

(a) $d(x, y) = 0 \iff x = y$.
(b) $d(x, y) = d(y, x)$.
(c) $d(x, z) \leq d(x, y) + d(y, z)$.

Proposition 2.0.2 (Subsets of a metric space)**.** *Let X be a metric space and $Y \subseteq X$. Then Y is also a metric space with the inherited metric.*

Definition 2.0.3 (Balls in a metric space)**.** Let (X, d) be a metric space. Let $x \in X$ and $r \in (0, \infty)$. Then $B_r(x) := \{y \in X : d(y, x) < r\}$ is an open ball (or just a ball), $D_r(x) := \{y \in X : d(y, x) \leq r\}$ is a closed ball (or a disc) and $S_r(x) := \{y \in X : d(y, x) = r\}$ is a sphere.

Proposition 2.0.4 (Metric associated with a norm / Linear metric)**.** *Let V be a vector space over $\mathbb{K}$ with a norm $\|\cdot\|$ on it. Define $d\colon V \times V \to [0, \infty)$ as*

$$d(x, y) := \|x - y\|.$$

Then d is a metric on X. This metric further satisfies the following for any $x, y, z \in V$ and any scalar α:

(a) $d(x + z, y + z) = d(x, y)$.
(b) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$.

Proposition 2.0.5 (Discrete metric)**.** *Let X be a set and define $\delta\colon X \times X \to [0, \infty)$ as*

$$\delta(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}.$$

Then δ is a metric on X.

Lemma 2.0.6. *Let $z_1, z_2, z_3 \in \mathbb{C}$. Then*
(a) $(1 + |z_1|^2)(1 + |z_2|^2) \geq |1 + z_1 z_2|^2$*, and*
(b) $(z_1 - z_2)(1 + |z_1|^2) = (z_1 - z_3)(1 + z_2 \overline{z_3}) + (z_3 - z_2)(1 + z_1 \overline{z_3})$.

Proposition 2.0.7 (Chord metric on $\mathbb{C}$)**.** *Define $d_c\colon \mathbb{C} \times \mathbb{C} \to [0, \infty)$ as*

$$d_c(z_1, z_2) := \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}.$$

Then d_c is a metric on $\mathbb{C}$.

Proposition 2.0.8 (Submetric space)**.** *Let (X, d) be a metric space and $Y \subseteq X$. Then the restriction of d on $Y \times Y$ is a metric on Y.*

2.1 Continuous functions and convergence in metric spaces

February 2, 2022

Definition 2.1.1 (Continuous functions on metric spaces)**.** Let (X_1, d_1) and (X_2, d_2) be metric spaces, $f\colon X_1 \to X_2$ and $x \in X_1$. Then f is called continuous at x iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $y \in X$, we have

$$d_1(y, x) < \delta \implies d_2(f(y), f(x)) < \varepsilon.$$

We call f to be continuous iff f is continuous at all $x \in X_1$.

Definition 2.1.2 (Cauchy sequences)**.** Let (X, d) be a metric space and $(x_i)_{i=i}^{\infty}$ be a sequence in X. Then it is said to be Cauchy iff for every $\varepsilon > 0$, there exists an $N \geq 1$ such that for every $i, j \geq N$, we have that $d(x_i, x_j) < \varepsilon$.

Definition 2.1.3 (Convergence of sequences)**.** Let (X, d) be a metric space and $x \in X$. Let $n \in \mathbb{Z}$ and $(y_i)_{i=n}^{\infty}$ be a sequence in X. Then $(y_i)_{i=n}^{\infty}$ is said to converge to x in (X, d), written $y_i \to x$, iff for every $\varepsilon > 0$, there exists an $N \geq n$ such that for all $n \geq N$, we have $d(y_n, x) < \varepsilon$.

Further, a sequence $(y_i)_{i=n}^{\infty}$ is said to be convergent, iff it converges to some $x \in X$.

Definition 2.1.4 (Complete metric spaces)**.** A metric space is called complete iff every Cauchy sequence is convergent.

Corollary 2.1.5.

(a) The convergent sequences converge to a unique limit.

(b) Convergent sequences are Cauchy.

Proposition 2.1.6 (Continuity equivalent to sequential continuity)**.** *Let X_1, X_2 be metric spaces, $f\colon X_1 \to X_2$ and $x \in X_1$. Then f is continuous at x $\iff$ for any sequence $(y_i)_{i=1}^{\infty}$, we have that*

$$y_i \to x \implies f(y_i) \to f(x).$$

Remark 2.1.7. Continuous functions in general don't preserve Cauchyness. Consider $f\colon (0, 1) \to \mathbb{R}$ given by $x \mapsto 1/x$ and consider the sequence $(1/n)_{n=1}^{\infty}$.

Theorem 2.1.8 (Composition of continuous functions)**.** *Let X_1, X_2, X_3 be metric spaces. Let $f\colon X_1 \to X_2$ be continuous at $x \in X_1$ and $g\colon X_2 \to X_3$ be continuous at $f(x) \in X_2$. Then $g \circ f\colon X_1 \to X_3$ is continuous at x.*

2.2 Uniform continuity

February 2, 2022

Definition 2.2.1 (Uniform continuity)**.** Let (X_1, d_1), (X_2, d_2) be metric spaces and $f\colon X_1 \to X_2$. Then f is uniformly continuous iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X_1$, we have that

$$d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \varepsilon.$$

Corollary 2.2.2. *Uniform continuity implies continuity for any function between metric spaces, and hence it preserves convergence.*

Proposition 2.2.3. *Uniformly continuous functions preserve Cauchy-ness.*

Theorem 2.2.4 (Prove this!). *Any real valued continuous function from a closed interval in $\mathbb{R}$ is uniformly continuous.*

Proposition 2.2.5 (Examples/non-exmaples of uniformly continuous functions).

(a) $x \mapsto x^2$ *is not uniformly continuous on* $\mathbb{R}$*, and neither is* $x \mapsto e^x$ *on* $\mathbb{R}$*. But they are uniformly continuous on any closed interval of* $\mathbb{R}$*.*

(b) Let (X, d) *be a metric space and* A *be a nonempty subset of* X*. Define* $d_A \colon X \to \mathbb{R}$ *as*

$$d_A(x) := \inf\{d(x, a) : a \in A\}.$$

Then d_A *is uniformly continuous (with the usual metric on* $\mathbb{R}$*).*

2.3 Equivalences in metric spaces

February 5, 2022

Definition 2.3.1 (Similarities and isometries). Let (X_i, d_i) be metric spaces for $i = 1, 2$ and $f \colon X_1 \to X_2$ be a bijection. Then f is called

(a) a similarity iff there exist $c_1, c_2 > 0$ such that for any $x, y \in X_1$, we have

$$c_1 d_1(x, y) \leq d_2(f(x), f(y)) \leq c_2 d_1(x, y), \text{ and}$$

(b) an isometry iff for all $x, y \in X_1$, we have

$$d_2(f(x), f(y)) = d_1(x, y).$$

Further, if such an f exists, we call the spaces similar, or respectively, isometric.

Proposition 2.3.2. *All the above relations are equivalences on any set of metric spaces.*

Corollary 2.3.3. *An isometry between two metric spaces is a similarity.*

Corollary 2.3.4. *The restrictions of similarities and isometries are similarities and isometries respectively.*

Proposition 2.3.5. *A similarity between two metric spaces is uniformly continuous.*

Corollary 2.3.6. *Similarities preserve Cauchy-ness and convergence of sequences.*

Definition 2.3.7 (Diameters of metric spaces)**.** Let (X, d) be a metric space. Then we define

$$\delta(X) := \sup\{d(x, y) : x, y \in X\}.$$

The metric is called bounded iff $\delta < +\infty$.

Corollary 2.3.8 (Diameter is an isometry-invariant)**.** *The diameters of isometric spaces are the same.*

Proposition 2.3.9 (Similarity $\not\Rightarrow$ isometry)**.** *Let (X, d) be a bounded metric space such that X has at least two elements. Define $d' \colon X \times X \to [0, \infty)$ as*

$$d'(x, y) := 2d(x, y).$$

Then d' is a metric on X. Now, the identity map on X is a similarity from (X, d) to (X, d'). However, (X, d) and (X, d') are not isometric.

Remark 2.3.10. This will not work for unbounded metric spaces like a normed vector space on $\mathbb{K}$. But we can still find bounded submetric spaces of V (like the unit sphere) and do something like above. But this says nothing about V, just about a bounded subset of it.

Proposition 2.3.11 (Metrics on finite product spaces)**.** *Let (X_i, d_i)'s be metric spaces for $i = 1, \ldots, n$. Let $p \in [1, \infty)$ and $X := \prod_{i=1}^{n} X_i$. Then we can define $D_p, D_\infty \colon X \times X \to [0, \infty)$ as*

$$D_p(x, y) := \Big(\sum_{i=1}^{n} d_i(x_i, y_i)^p\Big)^{1/p} \text{ and,}$$
$$D_\infty(x, y) := \max_{1 \le i \le n} d_i(x_i, y_i).$$

Then

(a) D_p and D_∞ are metrics on X,

(b) identity function is a similarity between (X, D_p) and (X, D_∞), and

(c) $x^{(i)} \to x$ in X (under D_p or D_∞) $\iff$ $x_j^{(i)} \xrightarrow{i} x_j$ (in (X_i, d_i)) for all $j = 1, \ldots, n$; hence the following are equivalent:

(i) (X, D_p) is complete.

(ii) (X, D_∞) *is complete.*

(iii) Each of the (X_i, d_i)*'s are complete.*

Corollary 2.3.12. *All the* l_p*-metrics are similar (identity function being a similarity) on* $\mathbb{K}^n$*.*

Proposition 2.3.13 (Convergence in l_p norms for $\mathbb{K}^n$)**.** *In* $\mathbb{K}^n$*,* $v_n \to v$ *with respect to the metric induced by the* l_p*-norm* $\iff$ $v_n \to v$ *with respect to the metric induced by the* l_∞*-norm.*

Proposition 2.3.14 (Isometry of l_1-, l_2-, l_∞-metrics on $\mathbb{R}^2$)**.** $(x, y) \mapsto (x + y, x - y)$ *is an isometry from* $\mathbb{R}^2$ *under* l_1*-metric to* $\mathbb{R}^2$ *under* l_∞*-metric. (In fact, it preserves) However, the* l_2*-metric is not isometric to either of* l_1*- or* l_∞*-metrics on* $\mathbb{R}^2$*. Prove the latter rigorously!*

Remark 2.3.15. *Prove this!* All norms on finite dimensional $\mathbb{K}^n$ over $\mathbb{K}$ are equivalent.

2.4 Some examples

February 2, 2022

Proposition 2.4.1 (Norms are continuous)**.** *A norm on a vector space* V *over* $\mathbb{K}$ *is continuous, under the metric induced by the norm on* V *and the usual metric on* $[0, \infty)$*.*

Proposition 2.4.2 (Metrics are continuous)**.** *Let* (X, d) *be a metric space and* $x_0 \in X$*. Then the function* $x \mapsto d(x, x_0)$ *is continuous, under the usual metric on* $[0, \infty)$*.*

Remark 2.4.3. We'll denote sequences in $\mathbb{K}^n$ as $\left(u^{(i)}\right)_{i=1}^{\infty}$, where $u^{(i)} = (u_1^{(i)}, \ldots, u_n^{(i)})$, for any $n \geq 1$.

In this subsection, we'll consider $\mathbb{K}^n$ with the linear metric of the l_p-norm for a fixed $p \in [1, \infty)$. Remember that all l_p-norms coincide for $\mathbb{K}$.

Theorem 2.4.4 (Convergence of sequences in $\mathbb{K}^n$)**.** *Let* $n \geq 1$ *and* $\left(u^{(i)}\right)_{i=1}^{\infty}$ *be a sequence in* $\mathbb{K}^n$ *and* $v \in \mathbb{K}^n$*. Then* $u^{(i)} \xrightarrow{i} v \iff u_j^{(i)} \xrightarrow{j} v_i$ *for all* $1 \leq i \leq n$*.*

Corollary 2.4.5 (Projections are continuous)**.** *Let* $n \geq 1$ *and* $1 \leq i \leq n$*. Then the function* $\pi_i \colon \mathbb{K}^n \to \mathbb{K}$ *defined by* $\pi_i(u) := u_i$ *is continuous.*

Corollary 2.4.6 (Continuity of functions in terms of projections)**.** *Let* (X, d) *be a metric space and* $f\colon X \to \mathbb{K}^n$ *for* $n \geq 1$. *Let* $x \in X$. *Then* f *is continuous at* $x \iff \pi_i \circ f\colon X \to \mathbb{K}$ *is continuous at* x *for all* $1 \leq i \leq m$.

Theorem 2.4.7 (Sums and products of convergent sequences in $\mathbb{K}$)**.** *Let* $u_i \to a$ *and* $v_i \to b$ *in* $\mathbb{K}$. *Then* $u_i + v_i \to a + b$ *and* $u_i v_i \to ab$ *in* $\mathbb{K}$.

Corollary 2.4.8 (Sums and products of continuous functions on $\mathbb{K}$)**.** *Let* $f, g\colon \mathbb{K} \to \mathbb{K}$ *be continuous at* $x \in \mathbb{K}$. *Then* $f+g$ *and* fg *(defined by pointwise operations) are continuous at* x.

Corollary 2.4.9 (Addition and scalar multiplication are continuous)**.** *Let* $n \geq 1$. *Then the usual vector space operations of addition and scalar multiplication on* $\mathbb{K}^n$ *(over* $\mathbb{K}$*) are continuous, upon identifying* $\mathbb{K}^m \times \mathbb{K}^k$ *with* $\mathbb{K}^{m+k}$ *(for which we have fixed the metric) for any* $m, k \geq 1$.

Corollary 2.4.10 (Polynomials are continuous)**.** *Let* $n \geq 1$ *and* $d \geq 0$. *Then, with multi-index notation, for any* c_α*'s, we have that the polynomial function* $p\colon \mathbb{K}^n \to \mathbb{K}$ *defined by* $p(x) := \sum_{\alpha \geq 0 : |\alpha| \leq d} c_\alpha x^\alpha$ *is continuous.*

Proposition 2.4.11 (Some complete metric spaces)**.** *The following are complete metric spaces:*

(a) $\mathbb{K}$ *under the usual metric.*
(b) $\mathbb{K}^n$ *under the* l_p *metric for any* $n \geq 1$ *and any* $p \in [0, \infty]$.
(c) $B(X; \mathbb{K})$ *with the usual metric.*

2.5 Completion of metric spaces

March 1, 2022

Definition 2.5.1 (Pseudo-metric)**.** A pseudo-metric d on X obeys all the conditions of being a metric except that $d(x, y)$ can be 0 even when $x \neq y$.

Proposition 2.5.2. *A pseudo-metric is a metric* $\iff$ *every singleton is closed in the pseudo-metric topology.*

Proposition 2.5.3 (Metric from a pseudo-metric)**.** *Let* (X, d) *be a pseudo-metric space. Then the relation* $\sim$ *on defined by* $x \sim y$ *iff* $d(x, y) = 0$ *is an equivalence relation. Further, there exists a metric* $\hat{d}\colon X \times X \to [0, \infty)$ *such that*

$$\hat{d}([x], [y]) = d(x, y).$$

Proposition 2.5.4 (Completion of metric spaces)**.** *Let (X, d) be a metric space. Let $\tilde{X}$ be the set of all Cauchy sequences in X (starting at some fixed index). We can define a $\tilde{d}\colon \tilde{X} \times \tilde{X} \to [0, \infty)$ as*

$$\tilde{d}\big((x_i)_i, (y_i)_i\big) := \lim_{i\to\infty} d(x_i, y_i).$$

Then $\tilde{d}$ is a pseudo-metric on $\tilde{X}$. We now define $(\hat{X}, \hat{d})$ to be the metric space induced by the pseudo-metric space $(\tilde{X}, \tilde{d})$ as in Proposition 2.5.3.

Then the $(\hat{X}, \hat{d})$ is complete and the function $\eta\colon X \to \hat{X}$ defined by

$$\eta(x) := [(x, x, \ldots)]$$

satisfies the following:

(a) *η is a distance-preserving embedding.*
(b) *Every open ball in $\hat{X}$ intersects with $\eta[X]$.*
(c) *X is complete $\implies$ η is an isometry.*

2.6 Topology of metric spaces

February 2, 2022

Lemma 2.6.1. *Union of unions can be written as a union.*

Lemma 2.6.2. *The intersection of two open balls in a metric space is a union of open balls.*

Part II

Topological spaces

Chapter 3

Main definitions

February 2, 2022

Definition 3.0.1 (Topological spaces)**.** Let X be a set and $\mathcal{T} \subseteq 2^X$. Then $\mathcal{T}$ is a topology on X iff the following hold:

(a) $X \in \mathcal{T}$.
(b) $\mathcal{T}$ is closed under arbitrary unions.
(c) $\mathcal{T}$ is closed under pairwise intersections.

We that $(X, \mathcal{T})$ is a topological space. We call the elements of $\mathcal{T}$ open sets, we call subsets of X whose complement (with respect to X) is open, closed.

Further, let $x \in X$ and A be a subset of X. Then A is called a neighborhood of x iff there exists an open U such that $x \in U \subseteq A$. We call A an open neighborhood of x if A is open also.

Remark 3.0.2. For finite $\mathcal{T}$, closure under pairwise unions is sufficient for (b).

Proposition 3.0.3 (Alternate definition of topology)**.** *Let X be a set and $\mathcal{T}, \mathcal{F} \subseteq 2^X$ such that $\mathcal{F} = \{X \setminus U : U \in \mathcal{T}\}$. Then $\mathcal{T}$ is a topology on X $\iff$ each of the following hold:*

(a) $\emptyset, X \in \mathcal{F}$.
(b) $\mathcal{F}$ *is closed under pairwise unions.*
(c) $\mathcal{F}$ *is closed under arbitrary nonempty intersections.*

Proposition 3.0.4 (Characterizing open and closed sets)**.** *Let A be a subset of a topological space X. Then*

(a) A is open $\iff$ *every point in A has an open neighborhood conatined in A, and*
(b) A is closed $\iff$ *every point contained in all of the closed sets containing A, is in A.*

Chapter 4

Bases and subbases

February 4, 2022

Theorem 4.0.1 (Intersection of topologies). *Let $\{\mathcal{T}_i\}_{i\in I}$ be a family of topologies on a set X. Then $\bigcap_{i\in I}\mathcal{T}_i$ is also a topology on X.*

Definition 4.0.2 (Topology generated by a set). Let X be a set and $S \subseteq X$. Then we call the smallest topology on X containing S the topology generated by S and denote it by $\mathcal{T}_S$.

Corollary 4.0.3. *Let X be a set. Let $A, B \subseteq 2^X$ and $\mathcal{T}$ be a topology on X. Then*

(a) $A \subseteq B \implies \mathcal{T}_A \subseteq \mathcal{T}_B$, and
(b) $A \subseteq \mathcal{T} \implies \mathcal{T}_A \subseteq \mathcal{T}$.

Definition 4.0.4 (Cover of a set). Let X be a set. Then $\mathcal{U} \subseteq 2^X$ is a cover of X iff $X = \bigcup \mathcal{U}$.

Definition 4.0.5 (Bases for topologies). Let X be a set. Then $\mathcal{B} \subseteq 2^X$ is called a base for a topology on X iff

(a) $\mathcal{B}$ is a cover for X, and
(b) for every $B_1, B_2 \in \mathcal{B}$ and for every $x \in B_1 \cap B_2$, there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ is a base for a topology on X, and $\mathcal{T}$ is a topology on X, then we say that $\mathcal{B}$ is a base for $\mathcal{T}$ iff $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$.

Theorem 4.0.6 (Characterization of the topology generated by a base). *Let X be a set. Let $\mathcal{B} \subseteq 2^X$ and $\mathcal{T}$ be a topology on X. Then the following are equivalent:*

(a) *$\mathcal{B}$ is a base for $\mathcal{T}$.*
(b) *$\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T} = \{\text{arbitrary unions of subfamilies of } \mathcal{B}\}$.*
(c) *$\mathcal{B} \subseteq \mathcal{T}$ and for every $U \in \mathcal{T}$ and every $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Definition 4.0.7 (Subbases)**.** Any set of subsets of X is called its subbase.

Remark 4.0.8. Every base is a subbase, but not conversely: Consider $\mathcal{S} := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$ for $\mathbb{R}$. (This is a base for the standard topology on $\mathbb{R}$ defined later.)

Theorem 4.0.9 (Characterization of the topology generated by a subbase)**.** *Let X be a set. Let $\mathcal{S} \subseteq 2^X$ and $\mathcal{T}$ be a topology on X. Let*

$$\mathcal{B}_{\mathcal{S}} := \{\text{finite nonempty intersections of subfamilies of } \mathcal{S}\} \cup \{X\}.$$

Then
(a) *$\mathcal{B}_{\mathcal{S}}$ is a base for $\mathcal{T}_{\mathcal{S}}$,*
(b) *$\mathcal{B}_{\mathcal{S}}$ is closed under finite intersections,*
(c) *$\mathcal{T}_{\mathcal{S}} = \{\text{arbitrary unions of finite nonempty intersections of subfamilies of } \mathcal{S} \cup \{X\}\}$, and*
(d) *if $\mathcal{S}$ is closed under finite nonempty intersections, then $\mathcal{B}_{\mathcal{S}} = \mathcal{S} \cup \{X\}$.*

Remark 4.0.10. Not all bases are closed under finite intersections: Consider the base of balls in general metric spaces.

Corollary 4.0.11. *Any topology is a base as well as a subbase for itself.*

Definition 4.0.12 (Base of a set)**.** Let X be a set and $\mathcal{B} \subseteq 2^X$. Then $\mathcal{B}$ is a base of X iff
(a) $\bigcup \mathcal{B} = X$, and
(b) for any $B_1, B_2 \in \mathcal{B}$, we have that $B_1 \cap B_2 = \bigcup \mathcal{C}$ for some $\mathcal{C} \subseteq \mathbb{B}$.

Remark 4.0.13. (b) above is equivalent to this: For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists a $C \in \mathcal{B}$ such that $x \in C \subseteq B_1 \cap B_2$.

Proposition 4.0.14 (Topology generated by a base)**.** *Let X be a set and $\mathcal{B}$ be a base for X. Let $\mathcal{T}$ be the set of all unions of subsets of $\mathcal{B}$. Then $\mathcal{T}$ is a topology on X.*

Proposition 4.0.15 (Some bases)**.**

(a) The set of singletons of a set X generate the discrete topology $(X, 2^X)$.

(b) The set of open balls of a metric space generate the metric topology.

(c) The following subsets of $\mathbb{R}$:

$$\big\{[a,b) : a,b \in \mathbb{R}\big\},$$
$$B \cup \{U \setminus K : U \in B\},$$

where $B := \{(a,b) : a,b \in \mathbb{R}\}$ *and* $K := \{1/n : n \geq 1\}$ *generate the* lower-limit *and* K*-topologies.*

Remark 4.0.16. For this section, we'll use these notations for the topologies on $\mathbb{R}$: $\mathcal{T}_{\text{std}}$ for the usual metric topology, $\mathcal{T}_{\text{l}}$ for the lower-limit topology, and $\mathcal{T}_K$ for the K-topology.

Remark 4.0.17. Unless stated otherwise, take $\mathbb{R}$ equipped with the standard (also called usual) topology.

Proposition 4.0.18 (Is $\mathcal{B}$ a base for $(X, \mathcal{T})$?)**.** *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subseteq \mathcal{T}$ such that for each $U \in \mathcal{T}$ and for each $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then $\mathcal{B}$ is a base for X which generates the topology $\mathcal{T}$ on X.*

Definition 4.0.19 (Subbases)**.** Let X be a set and $S \subseteq 2^X$. Then S is a subbase for X iff $\bigcup S = X$.

Proposition 4.0.20 (Subbases generate bases)**.** *Let S be a subbase for a set X and let $\mathcal{B}$ be the set of all nonempty finite intersections in S. Then $\mathcal{B}$ is a base for X.*

Corollary 4.0.21. *For a set X, let $\mathcal{S}$ be a subbase, $\mathcal{B}$ be a base and $\mathcal{T}$ be a topology. Then*

(a) $\mathcal{S} \subseteq \mathcal{B} \implies$ the topology generated by $\mathcal{S}$ is coarser than the topology generated by $\mathcal{B}$, and

(b) $\mathcal{S} \subseteq \mathcal{T} \implies$ the topology generated by $\mathcal{S}$ is coarser than $\mathcal{T}$.

Chapter 5

Comparing $\mathcal{T}_{\text{std}}$, $\mathcal{T}_{\text{l}}$ and $\mathcal{T}_K$

Definition 5.0.1 (Finer and coarser topologies)**.** Let $\mathcal{T}$, $\mathcal{T}'$ be topologies on a set X. Then $\mathcal{T}$ is called finer than $\mathcal{T}'$ iff $\mathcal{T} \supseteq \mathcal{T}'$, and is called strictly finer iff $\mathcal{T} \supsetneq \mathcal{T}$. Similarly for coarser.

We call them incomparable iff neither is a subset of the other.

Corollary 5.0.2. *Let $\mathcal{B}$, $\mathcal{B}'$ be bases for a set X. Then $\mathcal{B} \subseteq \mathcal{B}' \implies$ the topology generated by $\mathcal{B}$ is coarser than that generated by $\mathcal{B}'$.*

Proposition 5.0.3 (Comparing topologies)**.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X generated by bases $\mathcal{B}$ and $\mathcal{B}'$. Then the following are equivalent:*

(a) $\mathcal{T}$ is finer than $\mathcal{T}'$.
(b) $\mathcal{T} \supseteq \mathcal{B}'$.
(c) For any $x \in X$ and any $B' \in \mathcal{B}'$, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq B'$.

Remark 5.0.4. *Prove this!* The topologies generated by the open balls of l_2- and l_∞-norms on $\mathbb{R}^2$ are same.

Proposition 5.0.5 (Comparing $\mathcal{T}_{\text{std}}$, $\mathcal{T}_{\text{l}}$, $\mathcal{T}_K$)**.** *On $\mathbb{R}$, we have that $\mathcal{T}_{\text{l}}$ and $\mathcal{T}_K$ are both strictly finer that $\mathcal{T}_{\text{std}}$, whereas $\mathcal{T}_{\text{l}}$ and $\mathcal{T}_K$ are incomparable.*

Chapter 6

More examples of topological spaces

February 2, 2022

Proposition 6.0.1 (Discrete and indiscrete topologies)**.** *Let X be a set. Then 2^X and $\{\emptyset, X\}$ are topologies on X. Further, the discrete metric on any set induces the discrete topology.*

6.1 Metric topology

Proposition 6.1.1 (Metric and pseudo-metric topologies)**.** *Balls of any (pseudo-)metric space form a base for a topology, called the (pseudo-)metric topology.*

Definition 6.1.2 (Metrizable topologies)**.** A topological space for which there exists a metric which induces that topology, is called metrizable. Otherwise, it's called non-metrizable.

Proposition 6.1.3. *The discrete metric on any set induces the discrete topology.*

Proposition 6.1.4. *Any metric on a finite set induces the discrete topology.*

Corollary 6.1.5. *Any topology other than the discrete topology on a finite set having at least two elements is non-metrizable.*

6.2 Sierpiński topology

Definition 6.2.1 (Sierpiński point)**.** Let $(X, \mathcal{T})$ be a topological space. Then $x \in X$ is called a Sierpiński point for that space iff the only open set containing x is X.

Proposition 6.2.2 (Sierpiński topology)**.** *Let $X := \{0, 1\}$ and $\mathcal{S} := \{\emptyset, \{0\}, X\}$. Then $(X, \mathcal{S})$ is a topology with 1 being the Sierpiński point.*

Remark 6.2.3. *Prove that any topological space is completely determined by the set of continuous functions from it to $\mathcal{S}$.*

Proposition 6.2.4 (Sierpińskification)**.** *Let $(X, \mathcal{T})$ be a topological space and $\star \notin X$. Then $\mathcal{T} \cup \{X \cup \{\star\}\}$ is a topology on $X \cup \{\star\}$ with $\star$ being a Sierpiński point.*

6.3 *co$\mathcal{F}$* and *co$\mathcal{C}$*

Proposition 6.3.1 (Co-finite and co-countable topologies)**.** *Let X be a set and let*

$$\mathcal{T} := \{A \subseteq X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}.$$

Then $\mathcal{T}$ is a topology on X which coincides with the discrete topology if X is a finite set.

The same holds if finite is replaced with countable.

Further, we have the following two sets of equivalences:

(a) *(i) Co-finite and co-countable topologies on X coincide.*
(ii) X is finite.
(iii) Co-finite, co-countable and discrete topologies on X coincide.

(b) *(i) Co-countable and discrete topologies on X coincide.*
(ii) X is countable.

Proposition 6.3.2 (A characterization of co-finite topology)**.** *The co-finite topology on a set X is the smallest topology such that for every $x \in X$, the intersection of all the open neighborhoods of x is exactly $\{x\}$.*

The co-countable topology also satisfies this property.

Proposition 6.3.3 (A generator for co-finite topology)**.** *Let X be a set. Then the cover set*

$$\{X \setminus \{x\} : x \in X\}$$

generates the co-finite topology on X.

Further, for any $n \geq 1$, the cover set

$$\{U \subseteq X : |X \setminus U| = n\}$$

also generates the co-finite topology on X.

6.4 Order topology

Definition 6.4.1 (Open intervals and open rays). Let $\leq$ be a total order on a set X. Let $x, y \in X$. Then we set

(a) $(x, y) := \{z \in X : x < z < y\}$,
(b) $(x, +\infty) := \{z \in X : z > x\}$, and
(c) $(-\infty, x) := \{z \in X : z < x\}$.

Proposition 6.4.2 (Order topology). *Let $\leq$ be a total order on a non-singleton set X and $\mathcal{T}$ be the set of arbitrary unions of open rays or open intervals in X. Then $(X, \mathcal{T})$ is a topological space.*

Proposition 6.4.3 (Total orders which contain $\mathbb{Q}$). *Let $\leq$ be a total order on a set X having at least two elements such that for any $x, y \in X$, there exists a $z \in X$ such that $x < z < y$. Then there exist $a, b \in X$ such that (a, b) (with the inherited order) is similar to $\mathbb{Q}$ (with usual order on it).*

6.5 Topologies on $\mathbb{R}$

Remark 6.5.1. The metric topology induced by the usual metric on $\mathbb{R}$ is called the usual or the standard topology on $\mathbb{R}$.

Proposition 6.5.2. *Order topology on $\mathbb{R}$ coincides with the usual metric topology on it.*

Proposition 6.5.3 ($\mathcal{LR}$ and $\mathcal{RR}$ topologies). *On $\mathbb{R}$,*

$$\{(-\infty, a) : a \in \mathbb{R}\}, \text{ and}$$
$$\{(a, +\infty) : a \in \mathbb{R}\}$$

are subbases for the left and right ray topologies, and are not bases. Adjoining them with $\{\emptyset, \mathbb{R}\}$ gives the topologies. These are strictly coarser than the standard topology.

Proposition 6.5.4 (Semi-interval topology). *On* $\mathbb{R}$,

$$\big\{[a, b) : a, b \in \mathbb{R}, a < b\big\}$$

forms a base for a topology that is strictly finer than the usual topology.

6.6 Subspace topology

March 15, 2022

Proposition 6.6.1 (Subspace topology). *Let* $(X, \mathcal{T})$ *be a topological space and* $Y \subseteq X$*. Then*

$$\{U \cap Y : U \in \mathcal{T}\}$$

is a topology on Y*.*

Remark 6.6.2. We'll usually denote this by $\mathcal{T}|_Y$.

Corollary 6.6.3 (Transitivity of subspace topologies). *Let* $(X, \mathcal{T})$ *be a topological space and* $Z \subseteq Y \subseteq Z$*. Then*

$$\mathcal{T}|_Z = (\mathcal{T}|_Y)\big|_Z.$$

Remark 6.6.4. Unless stated otherwise, consider the subset of a topological space as equipped with the subspace topology.

Also, we'll talk of "closed" or "open subspaces of a topological space to mean that the subsets taken by themselves are open or closed in the parent topology.

Corollary 6.6.5. *Open (respectively closed) sets of open (respectively closed) subspaces are open (respectively closed) in the parent topology.*

Proposition 6.6.6 (Characterizing subspace topology by closed sets). *Let* X *be a topological space and* $Y \subseteq X$*. Then the closed sets in* Y *are precisely*

$$\{K \cap Y : K \text{ is closed in } X\}.$$

February 3, 2022

Proposition 6.6.7 (Subspace topology)**.** *Let $(X, \mathcal{T})$ be a topological space and $S \subseteq X$. Define*

$$\mathcal{T}' := \{S \cap U : U \in \mathcal{T}\}.$$

Then the $(S, \mathcal{T}')$ is a topology.

Remark 6.6.8. When talking about subsets of a topological space, consider them under the supspace topology, unless stated otherwise.

Remark 6.6.9. Sets that were not open (or were even closed) in the parent topology might become open in the subspace topology. Similarly for not closed (or even open). However, common subsets inherit openness.

Proposition 6.6.10 (A base for the subspace topology)**.** *Let $(X, \mathcal{T})$ be a topological space generated by a base $\mathcal{B}$ and $A \subseteq X$. Then $\{A \cap B : B \in \mathcal{B}\}$ is a base for the subspace topology for A.*

Proposition 6.6.11 (Characterization of subspace topology with closed sets)**.** *Let $\mathcal{F}$ be the set of closed sets of a topological space X. Let $Y \subseteq X$. Then the closed sets in the subspace topology are $\{Y \cap K : K \in \mathcal{F}\}$.*

Proposition 6.6.12. *The topology induced in a submetric space is exactly the subspace topology inherited from the metric topology of the parent space.*

Proposition 6.6.13. *The co-finite topology on a subset is the same as the subspace topology induced due to a co-finite topology.*

Proposition 6.6.14. *Let $(X, \mathcal{T})$ be a topological space and consider the subsets $Y_1, Y_2 \subseteq X$ with subspace topology. If $Y_1 \subseteq Y_2$, then Y_1's subspace topology as a subset of X is the same as its subspace topology as a subset of Y_2.*

6.7 Box topology

March 3, 2022

Lemma 6.7.1. *Let $\{A_{i,j}\}_{(i,j)\in I\times J}$ be a family of sets. Then*

$$\prod_{i\in I}\Big(\bigcup_{j\in J} A_{i,j}\Big) = \bigcup_{f\colon I\to J}\Big(\prod_{i\in I} A_{i,f(i)}\Big).$$

Proposition 6.7.2 (Box topology). *Let $(X_i, \mathcal{T}_i)_{i\in I}$ be topological spaces with bases $\mathcal{B}_i$'s. Define*

$$\mathfrak{B} := \left\{\prod_{i\in I} B_i : B_i \in \mathcal{B}_i\right\}, \text{ and}$$
$$\mathfrak{C} := \left\{\prod_{i\in I} U_i : U_i \in \mathcal{T}_i\right\}.$$

Then $\mathfrak{B}$ and $\mathfrak{C}$ are bases for topologies on $\prod_{i\in I} X_i$. Further, they generate the same topology, i.e., $\mathcal{T}_{\mathfrak{B}} = \mathcal{T}_{\mathfrak{C}}$.

6.8 Product topology

March 20, 022

Definition 6.8.1 (Product topology). Let $(X_i, \mathcal{T}_i)$ be topological spaces for $i \in I$. Then the topology on $\prod_i X_i$ generated by the subbase

$$\left\{\pi_i^{-1}(U_i) : i \in I, U_i \in \mathcal{T}_i\right\}$$

is called the product topology.

6.9 Product and box topologies

February 7, 2022

Proposition 6.9.1 (Product and box topologies). *Let I be a nonempty index set and $(X_i, \mathcal{T}_i)_{i\in I}$ be topological spaces. Define*

$$\mathcal{S}_\Pi := \bigcup_{i\in I}\left\{\pi_i^{-1}[U] : U \in \mathcal{T}_i\right\},$$
$$\mathcal{B}_\square := \left\{\prod_{i\in I} U_i : U_i \in \mathcal{T}_i\right\}.$$

Then S_Π and $\mathcal{B}_\square$ are respectively a subbase and a base for $\prod_{i\in I} X_i$.
Further,

(a) if I is a finite set then $\mathcal{T}_\Pi = \mathcal{T}_\square$, and

(b) if I is infinite, and none of X_i's is empty and for infinitely many i's, we have that $\mathcal{T}_i$'s are strictly finer than the respective indiscrete topologies, then $\mathcal{T}_\Pi \subsetneq \mathcal{T}_\square$.

Proposition 6.9.2 (Another characteristic for finite product topologies)**.** *Let $(X_i, \mathcal{T}_i)$ be topological spaces generated by bases $\mathcal{B}_i$ for $i = 1, 2$. Then $\mathcal{P} := \{B_1 \times B_2 : B_i \in \mathcal{B}_i\}$ is a base that generates the product (or box) topology on $X_1 \times X_2$.*

6.10 Miscellaneous

March 3, 2022

Proposition 6.10.1 (Zariski topology)**.** *Let $n \geq 1$. For each $f \in \mathbb{K}[x_1, \ldots, x_n]$, define*

$$U_f := \{x \in \mathbb{K} : f(x) \neq 0\}.$$

Then the set of all the U_f's forms a base for a topology on $\mathbb{K}^n$ such that $U_f \cap U_g = U_{fg}$.

Further, for $n = 1$, this topology coincides with the co-finite topology on $\mathbb{K}$.

Chapter 7

Continuous functions on topological spaces

February 2, 2022

Definition 7.0.1 (Continuous functions on topological spaces)**.** Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be topological spaces and $f\colon X_1 \to X_2$ and $x \in X_1$. Then f is continuous at x iff for every open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f[U] \subseteq V$.

f is called continuous iff it is continuous on all $x \in X_1$.

Remark 7.0.2. The identity function $\mathbb{R}_{\text{std}} \to \mathbb{R}$ is continuous when the codomain is taken under the standard topology, and is not continuous under the lower-limit topology.

Proposition 7.0.3 (Continuity compatible with metric spaces)**.** *Let X, Y be metric spaces and $f\colon X \to Y$ and $x \in X$. Then f is continuous at x in the sense of Definition 2.1.1 $\iff$ f is continuous at x in the sense of Definition 7.0.1.*

Proposition 7.0.4 (Characterizing continuity)**.** *Let X, Y be topological spaces. Let $f\colon X \to Y$ and $x \in X$. Then the following are equivalent:*

(a) f is continuous at x.
(b) Inverse images of open sets containing $f(x)$ are open.
(c) Inverse images of closed sets containing $f(x)$ are closed.
(d) For any subbase (and hence every base) of Y, the inverse images of subbase (respectively base) sets containing $f(x)$ are open.

Corollary 7.0.5 (Characterizing continuous functions). *Let X and Y be topological spaces. Let $f\colon X \to Y$ and $x \in X$. Then the following are equivalent:*

(a) f is continuous.
(b) Inverse images of open sets are open.
(c) Inverse images of closed sets are closed.
(d) For any subbase (and hence a base), the inverse images of subbse (respectively base) sets are open.

Remark 7.0.6. Let $f\colon X \to Y$ and $A \subseteq X$. Then by "a restriction of f to A", we mean any restriction with a codomain that contains $f[A]$.

But for by $f|_A$, we'll mean the restriction with the entire codomain Y.

Proposition 7.0.7 (Constructing continuous functions). *(a) Constant functions are continuous.*

(b) (Continuity of compositions) If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions between topological spaces such that f is continuous at $x \in X$ and g is continuous at $f(x)$, then $g \circ f\colon X \to Z$ is continuous at x.
(c) Any restriction of a continuous function is continuous.
(d) The inclusion function for a subspace is continuous.
(e) (Local formulation of continuity). *A function f is continuous $\iff$ for any subbase, all the restrictions of f to the subbase sets are continuous.*
(f) (Pasting lemma). *Let X, Y be topological spaces. Let A, B be both closed or both open in X such that $X = A \cap B$. Let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous such that they agree on $A \cap B$. Then the function $h\colon X \to Y$ defined by*

$$h(x) := \begin{cases} f(x), & x \in A \\ g(x), & x \in C \end{cases}$$

is continuous.
(g) Let X be a topological space and X_i's be subspaces of X such that $X = \bigcup_i X_i$. Let the topology on X be coherent with X_i's. Let $f\colon X \to Y$ for another topological space Y. Then f is continuous $\iff$ each $f|_{X_i}$ is continuous.
(h) Projections are continuous (for both, product as well as box topologies).
(i) (Continuity of $z \mapsto f_i(z)$ for box topology). *Let X_i's for $i \in I$ and Z be topological spaces. Let $f\colon Z \to \prod_i X_i$ be such that each $\pi_i \circ f\colon Z \to X_i$*

is continuous. Then f is continuous at z w.r.t. to the box topology $\implies$ each $\pi_i \circ f$ is continuous at z. The converse is true for finite I.

(j) (Continuous real-valued functions)**.** *The sum, product, negation a and reciprocation (for a function that never attains 0) of continuous real-valued functions are continuous.*

Remark 7.0.8.

(a) To show the necessity of A, B being both closed or both open in (f), consider $A = [0, 1]$ and $B = (1, \infty)$ for the subspace $X = [0, \infty)$ of $\mathbb{R}$. Then take any functions f and g such that h is discontinuous at 1 in the sense of Definition 2.1.1.
Another non-example: $f : \mathbb{Q} \to \mathbb{R}$ and $g : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ given by $f(x) = 1$ and $g(x) = 0$.

(b) To show the necessity of finite I for the converse in (i), consider $I = \mathbb{N}$ and each X_i's and Z to be $\mathbb{R}_{\text{std}}$ and $f_i(x) := ix$. Then $U := \prod_{i=0}^{\infty}(-1, 1)$ is open in the box topology, but $f^{-1}(U) = \{0\}$ is not open in Z.

Proposition 7.0.9 (Characterizing subspace topology)**.** *The subspace topology is the smallest topology that makes the inclusion continuous.*

Proposition 7.0.10 (Characterizing product topology)**.** *Let $(X_i, \mathcal{T}_i)$ be topological spaces for $i \in I$ and let $X := \prod_i X_i$ and $\mathcal{T}$ be a topology on X. Then the following are equivalent:*

(a) $\mathcal{T}$ is the product topology.

(b) $\mathcal{T}$ is the smallest topology such that all the projections $\pi_i : X \to X_i$ are continuous.

(c) $\mathcal{T}$ is the topology such that for any topological space Y and any function $f : Y \to X$, we have that f is continuous $\iff$ each $\pi_i \circ f : Y \to X_i$ is continuous.

7.1 Upper and lower semi-continuity

March 2, 2022

Definition 7.1.1 (Upper and lower semi-continuity)**.** Let X be a topological space. Let $f : X \to \mathbb{R}$. Then we say that

(a) f is upper semi-continuous at x iff for every $\varepsilon > 0$, there exists an open neighborhood U of x such that

$$y \in U \implies f(y) < f(x) + \varepsilon \text{ and,}$$

(b) f is called lower semi-continuous at x iff for every $\varepsilon > 0$, there exists an open neighborhood U of x such that

$$y \in U \implies f(x) - \varepsilon < f(y).$$

Further, if these hold for all points in X, then f is called upper/lower semi-continuous.

Proposition 7.1.2 (Characterizing upper and lower semi-continuities)**.** *Let X be a topological space. Let $f\colon X \to \mathbb{R}$ and $x \in X$. Then*

(a) f is upper semi-continuous at $x \iff f$ is continuous at x under the $\mathcal{LR}$ topology on $\mathbb{R}$,
(b) f is lower semi-continuous at $x \iff f$ is continuous at x under the $\mathcal{RR}$ topology on $\mathbb{R}$, and
(c) f is continuous at x with the usual topology on $\mathbb{R} \iff f$ is both upper and lower semi-continuous.

Proposition 7.1.3 (Openness and closedness via characteristic function)**.** *A set in a topological space is open (respectively closed) $\iff$ the corresponding characteristic function is lower (respectively upper) semi-continuous.*

Remark 7.1.4. Infimum and supremum for a family of (extended) real-valued functions are defined obviously.

Proposition 7.1.5. *Infimum (respectively supremum) of upper (respectively-lower) semi-continuous functions is upper (respectivelyupper) semi-continuous.*

Chapter 8

Convergence in topological spaces

February 20, 2022

Definition 8.0.1 (Convergence in topological spaces)**.** Let X be a topological space and $(x_i)_i$ be a sequence in X. Then it is said to converge to an $x \in X$, written $x_i \to x$ iff for every open neighborhood U of x, there exists an N such that for all $i \geq N$, we have that $x_i \in U$.

Further, $(x_i)_i$ is called convergent iff there exists an $x \in X$ such that $x_i \to x$.

Proposition 8.0.2 (Convergent sequences in discrete, *co*$\mathcal{F}$ and *co*$\mathcal{C}$ topologies)**.** *Let X be a set. Then we have the following:*

(a) Under the discrete topology*, the $x_i \to x \iff (x_i)_i$ becomes eventually equal to x.*

(b) Under the co-finite topology on infinite X*, we have that $x_i \to x \iff$ for every $k \geq 0$, and for any $y_1, \ldots, y_k \in \{x_i : i\} \setminus \{x\}$, there exists an N such that for every $i \geq N$, we have that $x_i \notin \{y_1, \ldots, y_k\}$.*

(c) Under the co-countable topology on uncountable X*, we have that $x_i \to x \iff (x_i)_i$ becomes eventually equal to x.*

Proposition 8.0.3 (Convergence compatible with metric spaces)**.** *Let X be a metric space and $(x_i)_i$ be a sequence in X. Then $(x_i)_i$ is convergent in the sense of Definition 2.1.3 $\iff$ it is convergent in the sense of Definition 8.0.1.*

Further, for any $x \in X$, we have that $x_i \to x$ in the sense of Definition 2.1.3 $\iff$ $x_i \to x$ in the sense of Definition 8.0.1.

Remark 8.0.4. Unlike in metric spaces, a sequence in a general topological space may converge to more than one point. For instance, every sequence converges to any Sierpiński point.

Proposition 8.0.5 (Sequential continuity in topological spaces)**.** *For topological spaces, continuity of a function at a point* $\implies$ *sequential continuity at that point.*

Further, if the domain space is metrizable, then the converse also holds.

Chapter 9

Homeomorphisms

Definition 9.0.1 (Homeomorphisms)**.** Let $(X_i, \mathcal{T}_i)$ be topological spaces for $i = 1, 2$ and $f\colon X_1 \to X_2$. Then f is a homeomorphism iff f is continuous and invertible with it inverse being also continuous.

Further, if there exists such a homeomorphism, then $(X_1, \mathcal{T}_1)$ is said to be homeomorphic to $(X_2, \mathcal{T}_2)$.

Remark 9.0.2. The continuity of inverse is important. Consider the identity function $\mathbb{R}_{\text{discrete}} \to \mathbb{R}_{\text{std}}$.

Definition 9.0.3 (Open and closed maps)**.** Let $(X_i, \mathcal{T}_i)$ be topological spaces for $i = 1, 2$. Then a function $f\colon X_1 \to X_2$ is called open iff it maps open sets to open sets, and it is called closed iff it maps closed sets to closed sets.

Corollary 9.0.4. *Let $(X_i, \mathcal{T}_i)$ be topological spaces for $i = 1, 2$ and $f\colon X_1 \to X_2$ be a bijection. Then the following are equivalent:*

(a) f^{-1} is continuous.
(b) f is open.
(c) f is closed.

Corollary 9.0.5. *"Being homeomorphic to" is an equivalence relation on any set of topological spaces.*

Corollary 9.0.6. *The set of all self-homeomorphisms on a topological space forms a group (under function composition).*

Proposition 9.0.7 (Restrictions of homeomorphisms are homeomorphisms)**.** *Let $(X_i, \mathcal{T}_i)$ be toplogical spaces for $i = 1, 2$. Let $f\colon X_1 \to X_2$ be a homeomorphism and $S \subseteq X_1$. Then f's restriction on S is a homeomorphism from S to $f[S]$ with respect to subspace topologies.*

Proposition 9.0.8 (Intervals are homeomorphic)**.** *Consider $\mathbb{R}$ with usual topology. Let $a < b$. Then the following pairs (under the subspace topology) are homeomorphic:*

(a) $(0, 1)$ *and* (a, b)*.*
(b) $(0, 1]$ *and* $(a, b]$*.*
(c) $[0, 1)$ *and* $[a, b)$*.*
(d) $[0, 1]$ *and* $[a, b]$*.*

Proposition 9.0.9 (*Prove this!*)**.** *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is a self-homeomorphism on $\mathbb{R}$ (under usual topology) $\iff$ f is surjective, continuous, and strictly monotonic.*

Proposition 9.0.10 (All lines in $\mathbb{R}^2$ are homeomorphic)**.** *Let $a, b, c \in \mathbb{R}$ such that one of a, b is nonzero. Then the set*

$$L := \{(x, y) \in \mathbb{R}^2 : ax + by = c\}$$

is homeomorphic to $\mathbb{R}$, with a possible homeomorphism given by

$$t \mapsto \begin{cases} \left(\frac{-bt+c}{a}, t\right), & a \neq 0 \\ \left(t, \frac{-at+c}{b}\right), & b \neq 0 \end{cases}.$$

9.1 Homeomorphisms on $\mathbb{K}^n$

February 3, 2022

Remark 9.1.1. For this subsection, fix a $p \in [0, \infty)$. For any $n \geq 1$,

(a) view $\mathbb{K}^n$ as the usual vector space over $\mathbb{K}$ equipped with the l_p-norm and consider it as a metric as well as a topological space with the induced metric and topology, and
(b) view any (proper) subsets of $\mathbb{K}^n$ as topological spaces under the subspace topology.

Remark 9.1.2. Also fix an $n \geq 1$ for this subsection. Then we define

$$\mathbb{B} := B_1(0),$$
$$\mathbb{D} := D_1(0),$$
$$\mathbb{S} := S_1(0),$$

where the right hand sides have the obvious usual meanings (for the metric we have fixed).

For $\alpha \in (0,\infty)^n$ and $x \in \mathbb{K}^n$, we'll define

$$\begin{aligned} R_\alpha(x) &:= \{y \in \mathbb{K}^n : d(y_i, x_i) < \alpha_i\}, \\ \overline{R_\alpha(x)} &:= \{y \in \mathbb{K}^n : d(y_i, x_i) \le \alpha_i\}, \\ \partial R_\alpha(x) &:= \{y \in \overline{R_\alpha(x)} : d(y_i, x_i) = \alpha_i \text{ for some } i\}. \end{aligned}$$

Also, we set

$$\begin{aligned} \mathbb{J} &:= R_{\mathbf{1}}(0), \\ \bar{\mathbb{J}} &:= \overline{R_{\mathbf{1}}(0)}, \\ \partial\mathbb{J} &:= \partial R_{\mathbf{1}}(0), \end{aligned}$$

where $\mathbf{1} := (1, \ldots, 1)$.

We'll denote $\|\cdot\|_p$ by $\|\cdot\|$, but we'll write $\|\cdot\|_\infty$ fully.

This is only for this subsection.

Proposition 9.1.3 (Vector addition and scalar multiplication are homeomorphisms)**.** *Let $z \in \mathbb{K}^n$ and $s \in \mathbb{K} \setminus \{0\}$. Define $T_z, M_s \colon \mathbb{K}^n \to \mathbb{K}^n$ as*

$$\begin{aligned} T_z(x) &:= x + z, \\ M_s(x) &:= sx. \end{aligned}$$

Then T_z and M_s are self-homeomorphisms on $\mathbb{K}^n$ with inverses given by T_{-z} and $M_{s^{-1}}$.

Corollary 9.1.4. *Let $x \in \mathbb{K}^n$ and $r > 0$. Then the following pairs are homeomorphic:*

(a) $\mathbb{B}$ and $B_r(x)$.
(b) $\mathbb{D}$ and $D_r(x)$.
(c) $\mathbb{S}$ and $S_r(x)$.

Proposition 9.1.5 ($\mathbb{K}^n$ homeomorphic to open balls)**.** *$\mathbb{B}$ is homeomorphic to $\mathbb{K}^n$. Two possible homeomorphisms $f, g \colon \mathbb{B} \to \mathbb{K}^n$ are given by*

$$f(x) = \frac{x}{1 - \|x\|}, \quad g(x) = \frac{x}{\sqrt{1 + \|x\|^2}},$$

with inverses

$$f^{-1}(y) = \frac{y}{1 + \|y\|}, \quad g^{-1}(y) = \frac{y}{1\sqrt{1 + \|y\|^2}}.$$

Proposition 9.1.6 (Linear transformations are continuous)**.** *Let A be an $n \times n$ matrix with entries in $\mathbb{K}$. Then $x \mapsto Ax^t$ is continuous. If A is further invertible, then this is a homeomorphism with the inverse given by $y \mapsto A^{-1}y^t$.*

Corollary 9.1.7. *Let $x \in \mathbb{K}^n$ and $\alpha \in (0, \infty)^n$. Then the following pairs are homeomorphic:*

(a) $\mathbb{J}$ and $R_\alpha(x)$.
(b) $\bar{\mathbb{J}}$ and $\overline{R_\alpha(x)}$.
(c) $\partial\mathbb{J}$ and $\partial R_\alpha(x)$.

Proposition 9.1.8 (Unit boxes are unit discs on l_∞-norm)**.** *$\mathbb{J}$, $\bar{\mathbb{J}}$, $\partial\mathbb{J}$ are precisely the unit open ball, the unit closed ball, and the unit sphere centered at the origin for the l_∞-norm on $\mathbb{K}^n$.*

Proposition 9.1.9 (Balls and boxes homeomorphic)**.** *Define $\phi\colon \mathbb{D} \to \bar{\mathbb{J}}$ as*

$$\phi(x) := \begin{cases} \frac{\|x\|}{\|x\|_\infty} x & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Then ϕ is a homeomorphism with inverse given by

$$\phi^{-1}(y) = \begin{cases} \frac{\|y\|_\infty}{\|y\|} y & y \neq 0, \\ 0 & y = 0 \end{cases}$$

Further, its restriction is a homeomorphism between $\mathbb{S}$ and $\partial\mathbb{J}$.

Remark 9.1.10. Construct the above homeomorphism in reverse: extend the homeomorphism from one between the boundaries.

Chapter 10

Topological equivalences of metric spaces

February 8, 2022

Definition 10.0.1. Two metric spaces are called topologically equivalent iff they are homeomorphic under the metric topology.

Proposition 10.0.2 (Similarity $\implies$ topological equivalence)**.** *Similarities between metric spaces are homeomorphisms.*

Proposition 10.0.3. *Let d, d' be two metrics on a set X. Then $\mathcal{T}(d) \subseteq \mathcal{T}(d') \iff$ for every $x \in X$ and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $B_{d',\delta}(x) \subseteq B_{d,\varepsilon}(x)$.*

Proposition 10.0.4 (All l_p-topologies equal to $\mathcal{T}_\square$ on $\mathbb{K}^n$)**.** *Let $n \geq 1$ and $p \in [1, \infty)$. Then $\mathcal{T}_p = \mathcal{T}_\infty = \mathcal{T}_\square$, where the box topology is for the n instances of $\mathbb{K}$, each with the usual topology.*

Proposition 10.0.5 (Topological equivalence $\not\Longrightarrow$ similarity)**.** *Let (X, d) be a metric space and define $D, D' \colon X \times X \to [0, \infty)$ as*

$$D(x,y) := \frac{d(x,y)}{1 + d(x,y)},$$
$$D'(x,y) := \min\{d(x,y), 1\}.$$

Then D, D' are bounded metrics on X. Hence, if d were unbounded, then (X, d) is not similar to either of (X, D) or (X, D'). However, $\mathcal{T}(D) = \mathcal{T}(d) = \mathcal{T}(D')$.

Proposition 10.0.6 (Another non-example)**.** *Let* (X_1, d_1) *and* (X_2, d_2) *be metric spaces and* $f\colon X \to Y$ *be a homeomorphism. Then the function* $\tilde{d}\colon X_1 \times X_1 \to [0, \infty)$ *defined by*

$$\tilde{d}(x, y) := d_2(f(x), f(y))$$

is a metric on X_1 *and* $\mathcal{T}(\tilde{d}) = \mathcal{T}(d_1)$*.*

Remark 10.0.7. $\mathcal{T}(d_1)$ and $\mathcal{T}(\tilde{d})$ needn't be similar: For $X = (-\pi/2, \pi/2)$ and $Y = \mathbb{R}$, both under the usual metrics, and with $f(x) = \tan x$, the Cauchy-ness of $(\pi/2 - 1/n)_{n=1}^{\infty}$ is not preserved.

Chapter 11

Subsets of a topological space

February 10, 2022

Definition 11.0.1 (Some terminology for sets and points in a topological space)**.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq A$ and $x \in X$. Then

(a) A is *open* iff $A \in \mathcal{T}$,
(b) A is *closed* iff $X \setminus A$ is open,
(c) A is an *open neighborhood* of x iff A is open and $x \in A$,
(d) A is a *neighborhood* of x iff A contains an open neighborhood of x,
(e) *interior* of A, denoted $\operatorname{int} A$ or $\mathring{A}$ is the union of all the open sets contained in A,
(f) x is a *closure point* of A iff every (open) neighborhood of x intersects with A,
(g) *closure* of A, denoted $\operatorname{cl} A$ or $\overline{A}$ is the set of all closure points of A,
(h) *boundary* of A, denoted ∂A, is the set $\overline{A} \setminus \mathring{A}$,
(i) x is a *limit or accumulation* point of A iff every (open) neighborhood of x intersects with $A \setminus \{x\}$ (or equivalently, any (open) neighborhood of x deleted $\{x\}$ intersects with A),
(j) A's *derived set*, denoted $\ell(A)$ or A', is the set of all of A's limit points,
(k) A is *dense* in X iff $\overline{A} = X$,
(l) A is *nowhere dense* in X iff $\operatorname{int} \operatorname{cl} A$ is empty,
(m) x is an *isolated point* of A iff there exists an (open) neighborhood of x that intersects A at only x,
(n) A is *isolated* iff for every point in A is isolated, and
(o) A is *discrete* iff A is isolated and closed.

Theorem 11.0.2 (Properties of closure)**.** *Let $(X, \mathcal{T})$ be a topological space* *and* $A, B \subseteq X$ *and* $(A_i)_{i\in I}$ *be a nonempty family of sets in* X*. Then*

(a) $\mathrm{cl}(\emptyset) = \emptyset$ *and* $\mathrm{cl}(X) = X$*,*
(b) $A \subseteq \mathrm{cl}(A)$*,*
(c) $A \subseteq B \implies \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$*,*
(d) A *is closed* $\implies \mathrm{cl}(A) = A$*,*
(e) $\mathrm{cl}(\mathrm{cl}(A)) = A$*,*
(f) A *is open and* A *does not intersect with* $B \implies A$ *does not intersect with* $\mathrm{cl}(B)$*,*
(g) $\mathrm{cl}(A)$ *is closed,*
(h) $\mathrm{cl}(A)$ *is the smallest closed set containing* A*,*
(i) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$*,*
(j) $\mathrm{cl}\big(\bigcup_{i\in I} A_i\big) \supseteq \bigcup_{i\in I} \mathrm{cl}(A_i)$*,*
(k) $\mathrm{cl}\big(\bigcap_{i\in I} A_i\big) \subseteq \bigcap_{i\in I} \mathrm{cl}(A_i)$*.*

Remark 11.0.3. For (j), consider $\mathbb{R} = \overline{\bigcup_{r\in\mathbb{Q}}\{r\}} \supsetneq \bigcup_{r\in\mathbb{Q}} \overline{\{r\}} = \mathbb{Q}$.

For (k), consider $\emptyset = \overline{\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})} \subsetneq \overline{\mathbb{Q}} \cap \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$.

Remark 11.0.4. When a same set can be viewed in two different topologies, we might use some notational tools to make clear the topology in which it is viewed.

Proposition 11.0.5 (Closures in subspaces)**.** *Let* Y *be a subspace of a topological space* X *and* $A \subseteq Y$*. Then*

(a) $\mathrm{cl}_Y(A) = Y \cap \mathrm{cl}(A)$*, and*
(b) Y *is closed* $\implies \mathrm{cl}_Y(A) = \mathrm{cl}(A)$*.*

Proposition 11.0.6 (Closure points using bases and subbases)**.** *Let* X *be a topological space with a base* $\mathcal{B}$ *and* $A \subseteq X$*. Let* $x \in A$*. Then* $a \in \overline{A} \iff$ *every base set containing* x *intersects with* A*.*

Similar result holds for any subbase of X*.*

Theorem 11.0.7 (Properties of interiors)**.** *Let* $(X, \mathcal{T})$ *be a topological space and* $A, B \subseteq X$ *and* $(A_i)_{i\in I}$ *be a nonempty family of sets in* X*. Then*

(a) $\mathrm{int}(\emptyset) = \emptyset$ *and* $\mathrm{int}(X) = X$*,*
(b) $\mathrm{int}(A) \subseteq A$*,*
(c) $A \subseteq B \implies \mathrm{int}(A) \subseteq \mathrm{int}(B)$*,*
(d) A *is open* $\implies \mathrm{int}(A) = A$*,*
(e) $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$*,*
(f) $\mathrm{int}(A)$ *is open,*

(g) $\operatorname{int}(A)$ *is the largest open set contained in* A*,*
(h) $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$*,*
(i) $\operatorname{int}\big(\bigcup_{i\in I} A_i\big) \supseteq \bigcup_{i\in I} \operatorname{int}(A_i)$*,*
(j) $\operatorname{int}\big(\bigcap_{i\in I} A_i\big) \subseteq \bigcap_{i\in I} \operatorname{int}(A_i)$*, and*
(k) A *and* B *are disjoint closed sets* $\implies \operatorname{int}(A \cup B) = \operatorname{int}(A) \cup \operatorname{int}(B)$*.*

Remark 11.0.8.

(a) For (j), consider $\emptyset = \operatorname{int}\big(\bigcap_{n\geq 1}(-1/n, 1/n)\big) \subsetneq \bigcap_{n\geq 1} \operatorname{int}\big((-1/n, 1/n)\big) = \{0\}$.
(b) For (i), consider $\mathbb{R} = \operatorname{int}\big(\bigcup_{x\in\mathbb{R}}\{x\}\big) \supsetneq \bigcup_{x\in\mathbb{R}} \operatorname{int}\big(\{x\}\big) = \emptyset$.
(c) For (k), consider
  (i) $A = [0, 1]$ and $B = [1, 2]$, and
  (ii) $A = [0, 1]$ and $B = (1, 2)$.

Proposition 11.0.9 (Interiors in subspaces)**.** *Let* Y *be a subspace of a topological space* X *and* $A \subseteq Y$*. Then*

(a) $\operatorname{int}_Y(A) \supseteq Y \cap \operatorname{int}(A)$*, and*
(b) Y *is open* $\implies \operatorname{int}_Y(A) = \operatorname{int}(A)$*.*

Remark 11.0.10. For (a), consider $Y = [0, 1]$ and $X = \mathbb{R}_{\text{std}}$.

Proposition 11.0.11 (Closures and interiors of complements)**.** *For any subset* A *of a topological space* X*,*

(a) $\operatorname{cl}(X \setminus A) = X \setminus \operatorname{int}(A)$*, and*
(b) $\operatorname{int}(X \setminus A) = X \setminus \operatorname{cl}(A)$*.*

Theorem 11.0.12 (Properties of boundary)**.** *Let* $(X, \mathcal{T})$ *be a topological space and* $A, B \subseteq X$ *and* $x \in X$*. Then*

(a) $\partial\emptyset = \emptyset$ *and* $\partial X = \emptyset$*,*
(b) ∂A *is closed,*
(c) $\partial(\partial A) \subseteq \partial A$ *with equality holding* $\iff \operatorname{int}(\partial A) = \emptyset$*,*
(d) $x \in \partial A \iff$ *every (open) neighborhood of* x *intersects with both* A *and* $X \setminus A$*,*
(e) $\partial A = \partial(X \setminus A)$*,*
(f) $\partial(A \cup B), \partial(A \cap B) \subseteq \partial A \cup \partial B$*, and*
(g) A *and* B *are disjoint, and both are either open, or both closed* $\implies \partial(A \cup B) = \partial A \cup \partial B$*.*

Remark 11.0.13. For (c), consider $\emptyset = \partial(\partial\mathbb{Q}) \subsetneq \partial\mathbb{Q} = \mathbb{R}$.

For (f), take $A = (0, 2)$ and $B = (1, 3)$.

Proposition 11.0.14. *For any subset A of a topological space, we have that* $\operatorname{int}(\partial(\partial A)) = \emptyset$ *so that* $\partial(\partial(\partial A)) = \partial(\partial A)$.

Also, $\partial A = \emptyset \implies A$ *is clopen.*

Theorem 11.0.15 (Properties of derived sets)**.** *Let $(X, \mathcal{T})$ be a topological space and $A, B \subseteq X$ and $(A_i)_{i\in I}$ be a nonempty family of sets in X. Then*

(a) $\ell(\emptyset) = \emptyset$,
(b) $\ell(X) = \{x \in X : \{x\}$ *is not open*$\}$,
(c) $A \subseteq B \implies \ell(A) \subseteq \ell(B)$,
(d) $\ell(A \cup B) = \ell(A) \cup \ell(B)$,
(e) $\ell\big(\bigcup_{i\in I} A_i\big) \supseteq \bigcup_{i\in I} \ell(A_i)$, *and*
(f) $\ell\big(\bigcap_{i\in I} A_i\big) \subseteq \bigcap_{i\in I} \ell(A_i)$.

Remark 11.0.16. $\ell(A)$ need not be open or closed and A, $\ell(A)$, $\ell(\ell(A))$ need not be comparable. Consider the indiscrete topology on $\{0, 1\}$. Then $\ell(\{0\}) = \{1\}$ and so on.

For (e), consider $\{0\} = \ell\big(\bigcup_{n\geq 1}\{1/n\}\big) \supsetneq \bigcup_{n\geq 1} \ell\big(\{1/n\}\big) = \emptyset$. For (f), consider $\emptyset = \ell\big(\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})\big) \subsetneq \ell(\mathbb{Q}) \cap \ell(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$.

Proposition 11.0.17. *Let X be a topological space and $A \subseteq X$. Then* $\operatorname{cl}(A) = A \cup \ell(A)$.

Also, any point in A is either its limit point or its isolated point, which are disjoint sets.

Lemma 11.0.18 (Characterizing dense sets)**.** *A set in a topological space is dense in it $\iff$ it intersects with every nonempty open set.*

Proposition 11.0.19 (Characterizing nowhere dense sets)**.** *Let A be a set in a topological space X. Then the following are equivalent:*

(a) $X \setminus \operatorname{cl}(A)$ *is dense in X.*
(b) $\operatorname{int}(\operatorname{cl}(A)) = \emptyset$.
(c) $\operatorname{cl}(A)$ *contains no nonempty open set.*
(d) Every nonempty open set in X has a nonempty open subset that is disjoint from $\operatorname{cl}(A)$.
(e) Every nonempty open set in X has a nonempty open subset that is disjoint from A.

Proposition 11.0.20 (Characterizations in metric spaces)**.** *Let A be a subset of a topological space X and $x \in X$. Then* *(a) there exists a sequence* $(x_i)_{i=n}^{\infty}$ *in* A *such that* $x_n \to x \implies x \in \mathrm{cl}(A)$*,*

(b) there exists a sequence $(x_i)_{i=n}^{\infty}$ *of distinct* x_n*'s in* A *such that* $x_n \to x \implies x \in \ell(A)$*,*

(c) every nonempty open set contains the closure of a nonempty open set that is disjoint from $A \implies A$ *is nowhere dense.*

Further, if X *is metrizable, then the converses of the above also hold.*

Remark 11.0.21. Necessity of metrizability in converses:

(a) For (a) and (b), consider $co\mathcal{C}$ on any uncountable set X: Take any uncountable $A \subsetneq X$. Then for any $x \in X \setminus A$, $x \in X = \mathrm{cl}(A)$, but no sequence in A converges to x. (See Propositions 8.0.2 and 11.1.8.)

(b) For (c), consider $co\mathcal{F}$ on any infinite set X: Take any nonempty finite $A \subseteq X$. Then it'll be nowhere dense, and closure of any nonempty open set will intersect A.

Proposition 11.0.22 (Characterizing continuous functions)**.** *Let* $(X_i, \mathcal{T}_i)$ *be topological spaces for* $i = 1, 2$ *and* $f\colon X_1 \to X_2$*. Then the following are equivalent:*

(a) f *is continuous.*

(b) $f[\mathrm{cl}(A)] \subseteq \mathrm{cl}(f[A])$ *for any* $A \subseteq X_1$*.*

(c) $f^{-1}[\mathrm{int}(B)] \subseteq \mathrm{int}(f^{-1}[B])$ *for any* $B \subseteq X_2$*.*

Proposition 11.0.23 (Kuratowski's closure axioms)**.** *Let* X *be a set and* c *be an operator on* 2^X *such that for all* $A, B \subseteq X$*, we have*

(a) $c(\emptyset) = \emptyset$*,*

(b) $A \subseteq c(A)$*,*

(c) $c(c(A)) = c(A)$*, and*

(d) $c(A \cup B) = c(A) \cup c(B)$*.*

Let $\mathcal{T}_c := \{U \subseteq X : c(X \setminus U) = X \setminus U\}$*. Then*

(a) for any $A \subseteq B \subseteq X$*, we have* $c(A) \subseteq c(B)$*, and*

(b) $\mathcal{T}_c$ *is a topology on* X *with* $\mathrm{cl}(A) = c(A)$ *for all* $A \subseteq X$*.*

Remark 11.0.24. Exactly parallel description can be given in terms of an interior operator i, with the same properties except that the first is replaced with $i(X) = X$.

11.1 Some examples

February 28, 2022

Proposition 11.1.1 (Closed sets in discrete, *co*$\mathcal{F}$ and *co*$\mathcal{C}$ topologies)**.** *Let X be a set. Then*

(a) Under discrete topology*, all the subsets are closed.*
(b) Under co-finite topology*, the closed sets are exactly the finite subsets.*
(c) Under co-countable topology*, the closed sets are exactly the countable subsets.*

Proposition 11.1.2 (Some open and closed sets in metric spaces)**.** *Let (X, d) be a metric space and $f\colon X \to \mathbb{R}$ be a continuous function under the usual metric on $\mathbb{R}$. Let K be closed (respectively open) in $\mathbb{R}$. Then $f^{-1}[K]$ is closed (respectively open) in X. In particular, disk? discs and spheres are closed, with open balls being open with*

$$\operatorname{cl}\big(B_\varepsilon(x)\big) \subseteq D_\varepsilon(x).$$

Other noteworthy sets are hyperplanes in $\mathbb{K}^n$ and the half-planes in $\mathbb{R}^n$, and their intersections.

Remark 11.1.3. In general, $\operatorname{cl}\big(B_\varepsilon(x)\big) \neq D_\varepsilon(x)$. Consider $B_1(x) = \{x\}$ in the discrete topology. Whereas its closure is itself, the corresponding disc is the entire space.

Proposition 11.1.4. *Every finite subset of a metric space is discrete.*

Proposition 11.1.5. *Consider $\mathbb{R}$ with the usual topology and a subset F of rationals, or of irrationals, be closed. Then F is nowhere dense in $\mathbb{R}$.*

Remark 11.1.6. There can be nonempty open sets that contain no closures of any nonempty set: Consider the set $\{0\}$ in the Sierpiński topology.

Remark 11.1.7. It can be that $X \setminus A$ is dense and $X \setminus \operatorname{cl}(A)$ is not: Consider $X = \mathbb{R}$ and $A = \mathbb{Q}$.

Proposition 11.1.8 (Closures and interiors in discrete, *co*$\mathcal{F}$ and *co*$\mathcal{C}$ topologies)**.** *Let X be a set and $A \subseteq X$. Then under any of the discrete, co$\mathcal{F}$ or co$\mathcal{C}$ topologies,*

(a) A is not open or $A = \emptyset \iff \operatorname{int}(A) = \emptyset$, and
(b) A is not closed or $A = X \iff \operatorname{cl}(A) = X$.

Proposition 11.1.9 (Closures, interiors, etc. under different topologies)**.** *Let* $A := \{1/n : n \geq 1\} \cup \{0\} \subseteq \mathbb{R}$*. Then we have the following:*

Topology	cl(A)	int(A)	∂A	$\ell(A)$
co-finite	$\mathbb{R}$	$\emptyset$	$\mathbb{R}$	$\mathbb{R}$
co-countable	A	$\emptyset$	A	$\emptyset$
lower ray	$[0,\infty)$	$\emptyset$	$[0,\infty)$	$(0,\infty)$

Lemma 11.1.10. *Prove this! A non-constant polynomial in* $n \geq 1$ *variables over* $\mathbb{K}^n$ *can't be uniformly zero over a nonempty open set of* $\mathbb{K}^n$*.*

Corollary 11.1.11. *For any non-constant* $p \in \mathbb{K}[x_1, \ldots, x_n]$ *for an* $n \geq 1$*, the set of zeros of* p *is nowhere dense in* $\mathbb{K}^n$*.*

11.2 F_σ, G_δ and meager sets

February 27, 2022

Definition 11.2.1 (F_σ, G_δ and meager sets)**.** Let X be a topological space and $A \subseteq X$. Then A is called

(a) an F_σ set[1] iff A is a countable union of closed sets,
(b) a G_δ set[2] iff A is a nonempty countable intersection of open sets,
(c) a meager or a first category set iff A is a countable union of nowhere dense sets, and
(d) a non-meager or a second category set iff A is not meager.

Proposition 11.2.2 (Some examples)**.**

(a) Countable sets are F_σ *in a metric space.*
(b) Any interval in $\mathbb{R}$ *is both,* F_σ *and* G_δ*.*

Proposition 11.2.3. *The inverse images of* F_σ *(respectively* G_δ*) sets are* F_δ *(respectively* G_δ*) under continuous functions between topological spaces.*

[1] French: fermé for F which means closed, and somme for σ, which means set union.

[2] German: Gebeit for G which means area or neigborhood, and δ for Durchshnitt used for set intersection.

Chapter 12

Cantor set

February 23, 2022

Remark 12.0.1. For this section:

(a) Let $\mathcal{S}$ be the set of all finite disjoint nonempty union of nonempty and non-singleton closed intervals.

(b) Let ϕ be the function on $\mathcal{S}$ well-defined (see Lemma 12.0.2) by

$$\phi\Big(\bigcup_{i=1}^{n}[a_i, b_i]\Big) := \bigcup_{i=1}^{n}\left(\left[a_i + \frac{b_i - a_i}{3}\right] \cup \left[b_i - \frac{b_i - a_i}{3}\right]\right)$$

for $a_i < b_i$. It follows that for each $A \in \mathcal{S}$, we have $\phi(A) \subsetneq \phi(A)$.

(c) Define $\mathfrak{C}\colon \{[a,b] : a < b\} \to 2^{\mathbb{R}}$ as follows: Given $a < b$, first define the following well-defined strictly descending sequence

$$J_0 := [a,b],$$
$$J_{n+1} := \phi(J_n).$$

Now, define

$$\mathfrak{C}([a,b]) := \bigcap_{i=1}^{\infty} J_n.$$

(d) For $[0,1]$, we'll denote the above sequence by I_n's and we'll set

$$C := \mathfrak{C}([0,1]).$$

Lemma 12.0.2 (Well-definedness of ϕ)**.** *Let $A \in \mathcal{S}$. Then there exist a unique $n \geq 1$ and unique $a_1 < b_1 < \cdots < a_n < b_n$ such that*

$$A = \bigcup_{i=1}^{n} [a_i, b_i].$$

Lemma 12.0.3. *Let $n \geq 1$ and let $\mathcal{J} := \bigcup_{i=1}^{n} J_i$ for nonempty and non-singleton closed intervals J_i's. Define $\{\mathcal{K}_i, \mathcal{L}_i\}$ to be the unique pair set of disjoint, nonempty and non-singleton closed intervals such that*

$$\phi(J_i) = \mathcal{K}_i \cup \mathcal{L}_i.$$

Then $\mathcal{X} := \bigcup_{i=1}^{n} \{\mathcal{K}_i, \mathcal{L}_i\}$ contains distinct disjoint sets and

$$\phi(\mathcal{J}) = \bigcup \mathcal{X}.$$

Proposition 12.0.4 (ϕ is linear)**.** *Let $k > 0$ and $r \in \mathbb{R}$, define $f, g \colon \mathbb{R} \to \mathbb{R}$ as*

$$f(x) := kx,$$
$$g(x) := x + a.$$

Then for any $I := [a, b]$ for $a < b$, we have that $f([a, b]), g([a, b]) \in \mathcal{S}$ and

$$\phi\big(f([a, b])\big) = f\big(\phi([a, b])\big),$$
$$\phi\big(g([a, b])\big) = g\big(\phi([a, b])\big).$$

Corollary 12.0.5 (Similarity between C and $\mathfrak{C}([a, b])$)**.** *Let $a < b$. Then the restriction of the similarity $f \colon \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) := a + (b - a)x$$

is a similarity from C to $\mathfrak{C}([a, b])$.

February 25, 2022

Lemma 12.0.6. *Let $N \geq 2$. Let $n \geq 1$ and $a_1, \ldots, a_n \in \mathbb{Z}$ such that each $|a_i| < N$. Let $S := \sum_{k=1}^{n} a_k / N^k$. Then*

(a) $|S| \leq 1 - 1/N^n$, and

(b) The sign of S is the same as that of the first nonzero a_i.

Proposition 12.0.7 (An explicit formula for I_n's)**.** *For $n \geq 1$, the set I_n is the disjoint union of the closed intervals $[\alpha, \alpha + 1/3^n]$ for $\alpha \in \left\{\sum_{k=1}^{n} a_k/3^k : a_k = 0, 2\right\}$. The intervals $[\alpha, \alpha + 1/3^n]$ are disjoint for distinct α's.*

Lemma 12.0.8. *Let $n \geq 1$ and $a_1 < b_1 < \cdots < a_n < b_n$. Let $\alpha < \beta$ and $(\alpha, \beta) \subseteq \bigcap_{i=1}^{n} [a_i, b_i]$. Then $(\alpha, \beta) \subseteq [a_i, b_i]$ for some i.*

Lemma 12.0.9. *Let $a < b$ and $\alpha < \beta$ such that $b - a < \beta - \alpha$. Then $(\alpha, \beta) \nsubseteq [a, b]$.*

Lemma 12.0.10 (Decimals in different bases)**.** *Prove this! Let $N \geq 1$. Let $a_1, a_2, \ldots \in \mathbb{N}$ such that each $a_i < N$. Then for each $\sum_{i=1}^{\infty} a_i/3^n$ converges, and we have that distinct sequences converge to distinct sums.*

Proposition 12.0.11 (Properties of C)**.**

(a) C is closed and bounded.
(b) Let J be one of the disjoint intervals of an I_n. Then $\mathfrak{C}(J) \subseteq C$.
(c) The endpoints of C, viz. 0 and 1, are in C.
(d) The endpoints of each of the disjoint intervals of each I_n are in C.
(e) C has no open intervals, and hence, being closed, it is nowhere dense.
(f) Every point of C is its limit point.
(g) C is uncountable.
(h) For every distinct $x, y \in C$, there exist disjoint closed subsets A, B of C such that $x \in A$ and $y \in B$ and $A \cup B = C$.
(i) Not rigorous yet! The length of C is 0.

Chapter 13

Metric trinity

February 27, 2022

Theorem 13.0.1 (Cantor's intersection theorem)**.** *Let X be a complete metric space and let $F_1 \supseteq F_2 \supseteq \cdots$ be a descending sequence of nonempty closed subsets such that $\delta(F_n) \to 0$ (in the usual topology in $\mathbb{R}$). Then $\cap_{i=1}^{\infty}$ is a singleton.*

Proposition 13.0.2 (Nested closed intervals and Bolzano-Weierstraß)**.** *Let X be a complete metric space where Bolzano-Weierstraß holds. Let $F_1 \supseteq F_2 \supseteq \cdots$ be a descending sequence of nonempty closed sets such that $\delta(F_n) \to D < \infty$. Then $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$.*

Proposition 13.0.3 (Some sequences of sets in $\mathbb{R}$)**.**

(a) With the usual metric on $\mathbb{R}$, the sets $A_n := [n, \infty)$ for $n \geq 0$ are closed with each $\delta(A_n) = +\infty$, and yet $\bigcap_{i=0}^{\infty} = \emptyset$.

(b) With the usual metric on $\mathbb{R}$, the sets $A_n := (0, 1/n)$ for $n \geq 1$ are open with each $\delta(A_n) = 1/n$ and yet $\bigcup_{i=1}^{\infty} = \emptyset$.

(c) With the discrete metric on $\mathbb{R}$, the sets in (b) become closed with $\delta(A_n) = 1$ for all $n \geq 1$. By Proposition 13.0.2, this means that Bolzano-Weierstraß is violated here.

Definition 13.0.4 (Contraction mappings)**.** Let X, Y be metric spaces. Then a function $f \colon X \to Y$ is called a contraction mapping iff there exists a $0 < c < 1$ such that for all $x, y \in X$, we have that $d(f(x), f(y)) \leq cd(x, y)$.

Theorem 13.0.5 (Banach's contraction mapping theorem)**.** *Every contraction mapping on a nonempty complete metric space has a unique fixed point.*

Definition 13.0.6 (Baire spaces)**.** A topological space X is called a Baire space iff X is of second category.

Lemma 13.0.7 (Diameters of balls and closures)**.** *Let X be a metric space. Then*

(a) for $A \subseteq X$,
$$\delta\big(\mathrm{cl}(A)\big) = \delta(A), \text{ and}$$
(b) for $\varepsilon > 0$ and $x \in X$,
$$\delta\big(B_\varepsilon(x)\big) \leq 2\varepsilon.$$

Remark 13.0.8. For (b), consider for discrete metric, in which $\delta(B_2(x)) = 1 < 4 = 2 \cdot 2$.

Theorem 13.0.9 (Baire's category theorem)**.** *Let X be a complete metric space and $A_1, A_2, \ldots \subseteq X$ be nowhere dense. Then $X \setminus \bigcup_{i=1}^\infty A_i$ is dense.*

Proposition 13.0.10 (Versions of BCT)**.** *Let X be a complete metric space. Then the following are equivalent:*

(a) Baire's category theorem.
(b) If $A_1, A_2, \ldots \subseteq X$ such that $X \setminus \bigcup_{i=1}^\infty A_i$ is not dense, then one of A_i's is not nowhere dense set.
(c) Countable nonempty intersection of open dense sets is dense.

Proposition 13.0.11 (Weak BCT)**.** *Every nonempty complete metric space is Baire space.*

Proposition 13.0.12 (Versions of weak BCT)**.** *Let X be a nonempty complete metric space. Then the following are equivalent:*

(a) Weak BCT.
(b) If $A_1, A_2, \ldots \subseteq X$ such that $X = \bigcup_{i=1}^\infty A_i$, then one of A_i's is not nowhere dense.
(c) Countable nonempty intersection of open dense sets is nonempty.

Remark 13.0.13. If A_i's are countably many nowhere dense sets, then $\bigcup_i A_i$ needn't be nowhere dense. Consider $A_r := \{r\}$ for $r \in \mathbb{Q}$.

Proposition 13.0.14 (Consequences of BCT)**.**

(a) $\mathbb{R} \setminus \mathbb{Q}$ is not F_σ in $\mathbb{R}$.
(b) $\mathbb{K}^n$ is not a union of the zero sets of countably many nonzero polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ for any $n \geq 1$.

13.1 Discontinuities in functions between metric spaces

Remark 13.1.1. For this subsection, we fix two metric spaces X and Y and a function $f\colon X \to Y$. Also, for any $UA \subseteq X$, we define

$$\omega(A) := \sup_{x,y\in A} d\big(f(x), f(y)\big),$$

and for any $x \in X$, we define

$$\omega(x) := \inf\{\omega(U) : U \text{ is an open neighborhood of } x\}.$$

We let $\mathfrak{D}$ to be the set of all the points in X where f is discontinuous.

Theorem 13.1.2 (Continuity via oscillations)**.** *Let $x \in X$. Then f is continuous at $x \iff \omega(x) = 0$.*

Theorem 13.1.3. *$\mathfrak{D}$ is F_σ in X.*

Corollary 13.1.4. *There can't be any function on $\mathbb{R}$ which is discontinuous on exactly $\mathbb{R} \setminus \mathbb{Q}$.*

Proposition 13.1.5 (The popcorn function)**.** *Define $g\colon \mathbb{R} \to \mathbb{R}$ by*

$$g(x) := \begin{cases} 0, & x \in \mathbb{R}\setminus\mathbb{Q} \\ 1/q, & x = p/q \text{ where } p, q \in \mathbb{Z}, q > 0, \gcd(p,q) = 1 \end{cases}.$$

Then

(a) g's left and right rand limits vanish for all $x \in \mathbb{R}$, and
(b) g is discontinuous on exactly $\mathbb{Q}$.

Chapter 14

Union of spaces: Coherent topology

March 15, 2022

Lemma 14.0.1. *Let $(X_i, \mathcal{T}_i)$'s be topological spaces and $X := \bigcup_i X_i$. Let*

$$\mathcal{T} := \{A \subseteq X : A \cap X_i \in \mathcal{T}_i\}.$$

Then

(a) $\mathcal{T}$ is a topology on X,
(b) the closed sets of $\mathcal{T}$ are precisely $\{B \subseteq X : B \cap X_i$ is closed in $X_i\}$, and
(c) $\mathcal{T}|_{X_i} \subseteq \mathcal{T}_i$ for each i.

Definition 14.0.2 (Coherent topology)**.** Let $(X_i, \mathcal{T}_i)$'s be topological spaces let $X := \bigcup_i X_i$. Then a topology $\mathcal{T}$ on X is called coherent with $(X_i, \mathcal{T}_i)$'s iff

(a) $\mathcal{T} = \{A \subseteq X : A \cap X_i \in \mathcal{T}_i\}$, and
(b) $\mathcal{T}_i = \mathcal{T}|_{X_i}$ for each i.

Characterize this!

Theorem 14.0.3. *Let $(X_i, \mathcal{T}_i)$'s be topological spaces and $X := \bigcup_i X_i$. Let*

(a) each $X_i \cap X_j$ be open (respectively closed) in both X_i and X_j, and
(b) $\mathcal{T}_i|_{X_i \cap X_j} = \mathcal{T}_j|_{X_i \cap X_j}$ for all i, j.

Then there is a (unique) topology $\mathcal{T}$ on X that is coherent with $(X_i, \mathcal{T}_i)$'s. In this topology, each X_i is an open (respectively closed) subspace in $(X, \mathcal{T})$.

Further, for the open case, the last property characterizes this topology.

Remark 14.0.4. To show that this doesn't characterize for the closed case, consider any space $(X, \mathcal{T})$ in which each singleton is closed. Then this characterization would imply that any such topology on X will be discrete. $\mathbb{R}_{\text{std}}$ provides a counterexample.

Corollary 14.0.5 (Disjoint union)**.** *Let $(X_i, \mathcal{T}_i)$'s be disjoint topological spaces and let $X := \bigcup_i X_i$. Then there exists a (unique) topology on X coherent with $(X_i, \mathcal{T}_i)$'s. In this topology, each X_i is clopen.*

Definition 14.0.6 (Locally finite sets)**.** Let X be a topological space and $\mathcal{C} \subseteq 2^X$. Then $\mathcal{C}$ is called locally finite at $x \in X$ iff there exists an (open) neighborhood U of X that intersects with only finitely many sets in $\mathcal{C}$.

If this happens for all $x \in X$, then $\mathcal{C}$ is called locally finite.

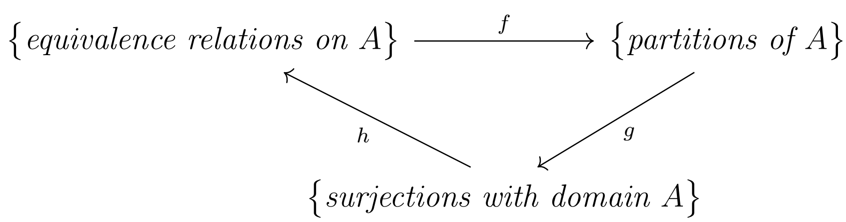
Proposition 14.0.7. *Let X_i's be closed subspaces of a topological space X such that $X = \bigcup_i X_i$ and X_i's form a locally finite set. Then the topology on X is coherent with the subspaces X_i's.*

Chapter 15

Quotient spaces

March 18, 2022

Proposition 15.0.1 (Making quotient sets in three equivalent ways)**.** *Let A be a set. Then there are one-to-one correspondences*



such that

$$h \circ g \circ f = \mathrm{id},$$
$$f \circ h \circ g = \mathrm{id},$$
$$g \circ h \circ f = \mathrm{id}.$$

These functions are given as follows:

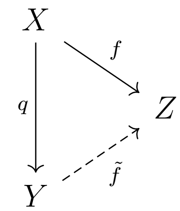
(a) *f assigns an equivalence relation $\sim$ on A to the set of its equivalence classes that form a partition of A.*

(b) *g assigns a partition $\mathcal{C}$ of A to the surjection $\mathfrak{f}\colon A \to \mathcal{C}$ defined so that $a \in \mathfrak{f}(a)$ for all $a \in A$.*

(c) *h assigns a surjection $\mathfrak{f}\colon A \to B$ to the equivalence relation $\sim$ on A defined by $a \sim b$ iff $\mathfrak{f}(a) = \mathfrak{f}(b)$.*

Lemma 15.0.2. *Let $q\colon X \to Y$ be a surjection and $f\colon X \to Z$ be a function. Then the following are equivalent:*

(a) $q(x_1) = q(x_2) \implies f(x_1) = f(x_2)$.
(b) *There exists a unique function* $\tilde{f}\colon Y \to Z$ *such that* $\tilde{f} \circ q = f$.



Further, the following pairs are equivalent:

(a) *(i)* $\tilde{f}$ *is injective.*
(ii) $f(x_1) = f(x_2) \implies q(x_1) = q(x_2)$.

(b) *(i)* $\tilde{f}$ *is surjective.*
(ii) f *is surjective.*

Remark 15.0.3. The surjectivity of q is needed to only guarantee uniqueness.

Corollary 15.0.4 (Bijection between $\mathbb{S}^1$ and $\mathbb{R}/\mathbb{Z}$)**.** *Let* $f\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ *be the canonical map taking* $\mathbb{R}$, $\mathbb{Z}$ *as additive groups. Let* $g\colon \mathbb{R} \to \mathbb{S}^1$ *defined by* $t \mapsto e^{2\pi i t}$. *Then there exists a bijection between* $\mathbb{R}/\mathbb{Z}$ *and* $\mathbb{S}^1$.

15.1 Group actions

March 19, 2022

Definition 15.1.1 (Group action)**.** Let G be a group and X be a set. Then a function $\circ\colon G \times X \to X$ is called an action of group G on AX iff the following hold:

(a) $1 \circ x = x$.
(b) $(gh) \circ x = g \circ (h \circ x)$.

Remark 15.1.2. We can call the above as the "left" group action, and similiarly can define right group actions.

By default, we'll talk of left actions, and when no confusion, we'll denote group action by juxtaposition.

We'll denote the group of permutations of a set X by $\sum(X)$.

Corollary 15.1.3 (Why called "actions"?)**.** *Let* G *be a group acting on a set* X *and* $g \in G$. *Then*

(a) $x \mapsto gx$ *is a permutation of* X*, and*
(b) *the function* $G \to \sum(X)$ *that maps* g *to the map* $x \mapsto gx$*, is a group homomorphism.*

Proposition 15.1.4 (Identifying a group action with a homomorphism $G \to \sum(X)$)**.** *Let* G *be a group and* X *be a set. Let* $\phi\colon G \to \sum(X)$ *be a group homomorphism. Then there exists a unique action of* G *on* X *such that it induces the same group homomorphism* ϕ*.*

Definition 15.1.5 (Orbits)**.** Let G be a group acting on a set X and $x \in X$. Then we call the set

$$G_x := \{gx : g \in G\}$$

the orbit of x.

Further, the set of all the orbits is called the orbit space.

Proposition 15.1.6. *Orbit space of a group action on a set* X *forms a partition of* X*.*

Remark 15.1.7. Abusing notation, we'll denote the orbit space by X/G.

Proposition 15.1.8 (Cosets as results of group actions)**.** *Let* H *be a subgroup of a group* G*. Then the restriction of the group operation to* $H \times G$ *is an action of* H *on* G*, the orbits of which are the right cosets of* H*.*

15.2 Quotient spaces

March 19, 2022

Proposition 15.2.1 (Quotient topology)**.** *Let* $(X, \mathcal{T})$ *be a topological space and* Y *be a set. Let* $q\colon X \to Y$ *be a function, and let*

$$\mathcal{T}' := \{V \subseteq Y : q^{-1}[V] \in \mathcal{T}\}.$$

Then

(a) $\mathcal{T}'$ *is a topology on* Y*,*
(b) q *becomes continuous with this topology on* Y*, and*
(c) $\mathcal{T}'$ *is the largest topology on* Y *for which* q *is continuous.*

Proposition 15.2.2 (Characterizing quotient topology)**.** *Let* X *be a topological space and* Y *be a set. Let* $f\colon X \to Y$*. Then the quotient topology on* Y *is the finest topology on* Y *such that* f *is continuous.*

Definition 15.2.3 (Quotient maps)**.** Let X be a topological space $q\colon X \to Y$ be a function for a set Y. Then we call q a quotient map iff Y is equipped with the topology

$$\mathcal{T}' := \{V \subseteq Y : q^{-1}[V] \in \mathcal{T}\}.$$

By Proposition 15.2.1, a quotient map is always continuous.

Remark 15.2.4. Note that we are *not* requiring q to be surjective.

Proposition 15.2.5. *Let X and Z be topological spaces and Y be a set. Consider a surjective quotient map $q\colon X \to Y$ and a continuous $f\colon X \to Z$ such that*

$$q(x_1) = q(x_2) \implies f(x_1) = f(x_2).$$

Then the unique function $\tilde{f}\colon Y \to Z$ such that $\tilde{f} \circ q = f$, is continuous.

Remark 15.2.6. Again the surjectivity only ensures uniqueness, and is redundant for continuity.

Proposition 15.2.7. *Let X and Z be topological spaces and Y be a set. Let $q\colon X \to Y$ be a quotient map and $f\colon Y \to Z$. Then $q \circ f$ is continuous $\iff$ f is continuous.*

Proposition 15.2.8. *A surjective open (or closed) function between topological spaces is a quotient map.*

15.3 Topological actions: A source of quotient spaces

March 20, 2022

Definition 15.3.1 (Topological actions and associated quotient maps)**.** A group action by a group G on a topological space X is called topological iff for every $g \in G$, the corresponding map $x \mapsto gx$ is a homeomorphism on X.

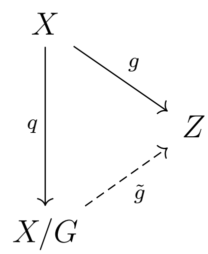
Since, G/X partitions X, we can associate a quotient.

Proposition 15.3.2. *The associated quotient map for a topological action is surjective and open.*

Proposition 15.3.3 (Homeomorphisms between X/G and another topological space)**.** *Let G be a group acting topologically on a topological space X and let q be its associated quotient map. Let Z be a topological space and $g\colon X \to Z$ be surjective and continuous such that for any $g \in G$ and any $x, y \in X$,*

$$x = gy \iff g(x) = g(y).$$

Let $\tilde{g}\colon X/G \to Z$ be the unique function such that $\tilde{g} \circ q = g$.



Then $\tilde{g}$ is a homeomorphism.

Corollary 15.3.4 ($\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$)**.** *Consider the equivalence relation $\sim$ on $\mathbb{R}$ defined by*

$$x \sim y \iff x - y \in \mathbb{Z}.$$

Then $\mathbb{R}/\sim$ endowed with the quotient topology is homeomorphic to $\mathbb{S}^1$.

Definition 15.3.5 (Real projective spaces)**.** Let ≥ 1 and $X := \mathbb{R}^{n+1} \setminus \{0\}$. Define an equivalence relation $\sim$ on X by

$$x \sim y \iff x = \lambda y \text{ for some } \lambda \in \mathbb{R} \setminus \{0\}.$$

Then $X/\sim$ endowed with the quotient topology is called the real n-dimensional projective space denoted by $\mathbb{P}^n$.