Organized results
Analysis I
Terence Tao

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Chapter 2

Starting at the beginning: the natural numbers

2.1 The Peano axioms

April 23, 2021

Axiom 2.1 (Natural numbers)**.** *Natural numbers are objects.*

Remark 2.1.1. The domain of the function symbol " ++" is natural numbers so that $n{+}{+}$ is an object for each natural number n. We also have a constant symbol 0, which is hence an object.

Axiom 2.2 (Peano axioms)**.** *(i)* 0 *is a natural number.*

(ii) For each natural number n, we have that $n{+}{+}$ is a natural number.

(iii) For any natural number n, we have $n{+}{+} \neq 0$.

(iv) For any natural numbers m and n, we have $m{+}{+} = n{+}{+} \implies m = n$.

(v) (Principle of mathematical induction). *Let $P(n)$ be a property pertaining to any natural number n such that $P(0)$ holds and for each natural number n, we have $P(n{+}{+})$ whenever $P(n)$ holds. Then $P(n)$ holds for each natural number n.*

Remark 2.1.2. We don't define "++" or equality for naturals. They are primitive concepts which are assumed to obey axiom of substitution. Similar remarks when axiomatizing other relations.

Remark 2.1.3. "++" can be replaced by a binary relation symbol, but at an expense of an additional uniqueness axiom.

May 2, 2021

Remark 2.1.4. For the use of $\mathbb{N}$, one needs sets. We also need functions (which also rest on first having sets) in what follows, but there is a path possible, intertwining Chapters 2 and 3 so that there is no circularity. The only things in Chapter 3 that rely on the properties of naturals that we'll derive here will be the stuff ordered n-tuples onward, and that will not be used here. Hence, everything (not strictly) before ordered n-tuples can be used here.

Proposition 2.1.5 (Recursive definitions)**.** *Let $c \in \mathbb{N}$ and $\{f_n\}_{n\in\mathbb{N}}$ be a family of functions such that $f_n\colon \mathbb{N} \to \mathbb{N}$ for each $n \in \mathbb{N}$. Then there exists a unique function a such that $a\colon \mathbb{N} \to \mathbb{N}$ so that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each $n \in \mathbb{N}$.*

Remark 2.1.6. Proved in Chapter 3.

2.2 [Custom]

May 7, 2021

Lemma 2.2.1 (Increment function)**.** *There exists a unique function f such that $f\colon \mathbb{N} \to \mathbb{N}$ and $f(n) = n{++}$ for each $n \in \mathbb{N}$.*

Remark 2.2.2. Hence the primitive ++ can be viewed as a function.

Lemma 2.2.3 (Addition on $\mathbb{N}$)**.** *Let $m \in \mathbb{N}$. Then there exists a unique function f such that $f\colon \mathbb{N} \to \mathbb{N}$ so that $f(0) = m$ and $f(n{++}) = f(n){++}$ for each $n \in \mathbb{N}$.*

Remark 2.2.4. This allow to denote $f(n)$ by $n + m$ for each $n \in \mathbb{N}$. Axiom of substitution satisfied.

Corollary 2.2.5. *Let $m, n \in \mathbb{N}$. Then $0 + m = 0$ and $(n{++}) + m = (n + m){++}$.*

Lemma 2.2.6. *Let $n \in \mathbb{N}$. Then $n + 0 = n$.*

Remark 2.2.7. We set $1 \coloneqq 0{++}$ and so on. We'll not mention such remarks again.

Lemma 2.2.8. *Let* $m, n \in \mathbb{N}$*. Then* $m + (n{+}{+}) = (m + n){+}{+}$*.*

Corollary 2.2.9. *Let* $n \in \mathbb{N}$*. Then* $n{+}{+} = n + 1$*.*

Lemma 2.2.10 (Multiplication on $\mathbb{N}$)**.** *Let* $m \in \mathbb{N}$*. Then there exists a unique function* f *such that* $f : \mathbb{N} \to \mathbb{N}$ *so that* $f(0) = 0$ *and* $f(n + 1) = f(n) + m$ *for each* $n \in \mathbb{N}$*.*

Remark 2.2.11. This allows to denote $f(n)$ by nm for each $n \in \mathbb{N}$. Axiom of substitution obeyed.

Corollary 2.2.12. *Let* $m, n \in \mathbb{N}$*. Then* $0m = 0$ *and* $(n + 1)m = (nm) + m$*.*

Remark 2.2.13. We'll assume the usual precedence of operations, omitting some parentheses.

Lemma 2.2.14. *Let* $n \in \mathbb{N}$*. Then* $n0 = 0$*.*

Lemma 2.2.15. *Let* $m, n \in \mathbb{N}$*. Then* $m(n + 1) = mn + m$*.*

Proposition 2.2.16. *Let* $a, b, c \in \mathbb{N}$*. Then*

$$\begin{aligned}
a + b &= b + a, \\
(a + b) + c &= a + (b + c), \\
a + 0 = 0 + a &= a, \\
ab &= ba, \\
(ab)c &= a(bc), \\
a1 = 1a &= a, \\
a(b + c) &= ab + ac, \text{ and} \\
(a + b)c &= ac + bc.
\end{aligned}$$

Remark 2.2.17. Prove distributivity before associativity of multiplication.

Proposition 2.2.18 ($\mathbb{N}$ has no zero addends)**.** *Let* $a, b \in \mathbb{N}$ *such that* $a + b = 0$*. Then* $a = b = 0$*.*

Proposition 2.2.19 ($\mathbb{N}$ has no zero divisors)**.** *Let* $a, b \in \mathbb{N}$ *such that* $ab = 0$*. Then* $a = 0$ *or* $b = 0$*.*

Proposition 2.2.20 (Cancellation law for addition on $\mathbb{N}$)**.** *Let* $a, b, c \in \mathbb{N}$*. Then* $a + c = b + c \implies a = b$*.*

Definition 2.2.21 (Positive naturals)**.** An object n is called a positive natural iff $n \in \mathbb{N}$ and $n \neq 0$.

Remark 2.2.22. Axiom of substitution obeyed.

Corollary 2.2.23 (Sums and products of positives are positive)**.** *Let a, b be positive naturals. Then $a + b$ and ab are positive naturals.*

Proposition 2.2.24. *Let a be a positive natural. Then there exists a unique $b \in \mathbb{N}$ such that $a = b + 1$.*

Definition 2.2.25 (Order on $\mathbb{N}$)**.** For any objects m, n, we write

(i) "$m > n$", or "$n < m$", iff $n \in \mathbb{N}$ and $m = n + a$ for some positive natural a, and
(ii) "$m \geq n$", or "$n \leq m$", iff $n \in \mathbb{N}$ and $m = n + a$ for some natural a.

Remark 2.2.26. Axiom of substitution satisfied.

Corollary 2.2.27 (Characterizing "$\geq$" and "$>$")**.** *Let $a, b \in \mathbb{N}$. Then*

(i) $a \geq b \iff a > b$ *or* $a = b$, *and*
(ii) $a > b \iff a \geq b$ *and* $a \neq b$.

Lemma 2.2.28. *Let $a, b \in \mathbb{N}$. Then*

(i) $a + 1 > a$, *and*
(ii) $a < b \iff a + 1 \leq b$.

Proposition 2.2.29 (Trichotomy of order on $\mathbb{N}$)**.** *Let $a, b \in \mathbb{N}$. Then exactly one of the these holds: $a < b$, $a = b$, or $a > b$.*

Corollary 2.2.30 (Order properties on $\mathbb{N}$)**.** *Let $a, b, c \in \mathbb{N}$. Then*

(i) (transitivity) $a < b$ *and* $b < c \implies a < c$,
(ii) (addition preserves order) $a < b \implies a + c < b + c$, *and*
(iii) (multiplication by positives preserves order) $a < b$ *and* c *is positive* $\implies ac < bc$.

Corollary 2.2.31 (Cancellation law for multiplication on $\mathbb{N}$)**.** *Let $a, b, c \in \mathbb{N}$ such that $c \neq 0$ and $ac = bc$. Then $a = b$.*

Definition 2.2.32 (Even and odd naturals)**.** An object m is called

(i) an even natural iff there exists an $n \in \mathbb{N}$ such that $m = 2n$, and
(ii) an odd natural iff there exists an $n \in \mathbb{N}$ such that $m = 2n + 1$.

Remark 2.2.33. Axiom of substitution obeyed.

Proposition 2.2.34 (Properties of odds and evens)**.** *Let $n \in \mathbb{N}$. Then*

(i) n is an odd natural $\implies$ $n+1$ is an even natural,
(ii) n is an even natural $\implies$ $n+1$ is an odd natural,
(iii) n is an odd natural or an even natural, but not both.

Remark 2.2.35. For $m \in \mathbb{N}$ and a property $P(i)$ pertaining to any $i \in \mathbb{N}$, we'll write "$P(i)$ holds for each $i \leq m$" to mean "for each natural number i we have that $i \leq m \implies P(i)$". Similar remarks for the existential quantifier and for other possible generalizations.

Proposition 2.2.36 (Equivalent forms of induction)**.** *The following are all equivalent to the principle of induction:*

(i) (Induction from base case m_0)**.** *Let $m_0 \in \mathbb{N}$ and $P(m)$ be a property pertaining to any $m \in \mathbb{N}$ such that $P(m_0)$ holds and for each $m \in \mathbb{N}$, we have that $P(m) \implies P(m+1)$. Then $P(m)$ holds for each $m \geq m_0$.*
(ii) (Strong induction)**.** *Let $m_0 \in \mathbb{N}$, and $P(m)$ be a property pertaining to any $m \in \mathbb{N}$ such that for each $m \geq m_0$, we have that $P(m)$ follows if $P(m')$ holds for each $0 \leq m' < m$. Then $P(m)$ holds for each $m \geq m_0$.*
(iii) (Backwards induction)**.** *Let $m_0 \in \mathbb{N}$ and $P(m)$ be a property pertaining to any $m \in \mathbb{N}$ such that $P(m_0)$ holds and for each $m \in \mathbb{N}$, we have that $P(m+1) \implies P(m)$. Then $P(m)$ holds for each $m \leq m_0$.*
(iv) (Principle of infinite descent)**.** *Let $P(n)$ be a property pertaining to any $n \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, we have that $P(n) \implies$ there exists an $m \in \mathbb{N}$ such that $P(m)$ holds and $m < n$. Then $P(n)$ is false for all $n \in \mathbb{N}$.*
(v) (Well ordering principle)**.** *Let S be a set such that $S \subseteq \mathbb{N}$ and $S \neq \emptyset$. Then there exists a unique $n \in S$ such that $n \leq x$ for each $x \in S$.*

Proposition 2.2.37 (Euclid's division lemma)**.** *Let $n, q \in \mathbb{N}$ such that q is a positive natural. Then there exist unique $m, r \in \mathbb{N}$ such that $n = mq + r$ and $0 \leq r < q$.*

Lemma 2.2.38 (Exponentiation on $\mathbb{N}$)**.** *Let $m \in \mathbb{N}$. Then there exists a unique function f such that $f\colon \mathbb{N} \to \mathbb{N}$ so that $f(0) = 0$ and $f(n+1) = f(n)m$ for each $n \in \mathbb{N}$.*

Remark 2.2.39. This allows to denote $f(n)$ by m^n for each $n \in \mathbb{N}$. Axiom of substitution obeyed.

Chapter 3

Set theory

3.1 Fundmentals

May 8, 2021

Axiom 3.1 (Sets)**.** *Sets are objects.*

Axiom 3.2 ("Element of" relation)**.** *For any objects x, A, we have that $x \in A \implies A$ is a set.*

Axiom 3.3 (Equality of sets)**.** *Let A, B be sets and for any object x, let $x \in A \iff x \in B$. Then $A = B$.*

Axiom 3.4 (Empty set)**.** *There exists a set A such that for any object x, we have $x \notin A$.*

Remark 3.1.1. It follows that existence above can be strengthened to unique existence, allowing to denote A by $\emptyset$.

Remark 3.1.2. Uniqueness of objects is implicitly assumed to be uniqueness up to equality for that class of objects.

Definition 3.1.3. (i) An object A is called nonempty iff $A \neq \emptyset$.

(ii) For an object A, we write "A is a nonempty set" iff A is nonempty and A is a set.

Remark 3.1.4. Axiom of substitution obeyed.

Lemma 3.1.5 (Single choice)**.** *Let A be a nonempty set. Then there exists an object $x \in A$.*

Axiom 3.5 (Singletons and pair sets). *Let a, b be objects. Then*

(i) (singletons) *there exists a set X such that for any object x, we have $x \in X \iff x = a$, and*
(ii) (pair sets) *there exists a set Y such that for any object y, we have $y \in Y \iff y = a$ or $y = b$.*

Remark 3.1.6. Existence can again be strengthened to unique existence in both of the above, allowing to denote X and Y respectively as $\{a\}$ and $\{a, b\}$. This also obeys axiom of substitution.

Remark 3.1.7. (ii) implies (i).

Axiom 3.6 (Pairwise unions). *Let A, B be sets. Then there exists a set X such that for any object x, we have $x \in X \iff x \in A$ or $x \in B$.*

Remark 3.1.8. Existence again can be strengthened to unique existence, allowing to denote X by $A \cup B$ This also obeys axiom of substitution.

Lemma 3.1.9. *Let a, b be objects. Then $\{a\} \cup \{b\} = \{a, b\}$.*

Remark 3.1.10. Axiom 3.5 (i) (singletons) and Axiom 3.6 (pairwise unions) together imply Axiom 3.5 (ii) (pair sets).

Definition 3.1.11 (Subsets). For any objects A, B, we write

(i) "$A \subseteq B$", or "$B \supseteq A$", iff A and B are sets and for any object x, we have $x \in A \implies x \in B$, and
(ii) "$A \subsetneq B$", or "$B \supsetneq A$", iff $A \subseteq B$ and $A \neq B$.

Remark 3.1.12. Obeys axiom of substitution.

Corollary 3.1.13. *Let A be a set. Then $\emptyset \subseteq A$.*

Lemma 3.1.14. *Let A, B be sets and $A \subsetneq B$. Then there exists an object x such that $x \in B$, but $x \notin A$.*

Proposition 3.1.15 (Sets are partially ordered by $\subseteq$). *Let A, B, C be sets. Then*

(i) (reflexive) $A \subseteq A$,
(ii) (transitive) $A \subseteq B$ *and* $B \subseteq C \implies A \subseteq C$, *and*
(iii) (anti-symmetric) $A \subseteq B$ *and* $B \subseteq A \implies A = B$.

Remark 3.1.16. Some notes on properties:

(i) Properties are formed by constant symbols, function symbols, atomic relations, and logical connectives and quantifiers.
(ii) Suppose $P(x)$ is a predicate symbol, and A is a set. We then write "$P(x)$ pertains to any $x \in A$" iff for any object x, we have $P(x) \implies x \in A$. Similar comments when "pertains to" is followed by some objects of specified types, possibly mentioned to satisfy some other conditions.

Axiom 3.7 (Specification)**.** *Let A be a set and $P(x)$ be a property pertaining to any $x \in A$. Then there exists a set X such that for any object x, we have $x \in X \iff x \in A$ and $P(x)$ holds.*

Remark 3.1.17. Existence above can be again strengthened to unique existence, allowing to denote X by $\{x \in A : P(x)\}$. Axiom of substitution satisfied.

Remark 3.1.18. If we can show that a set exists (for instance, $\{0\}$ exists because of Axioms 2.1 (natural numbers) and 3.5 (singletons and pair sets); one can also use Axiom 3.9 (infinity)), then Axiom 3.7 (specification) implies Axiom 3.4 (empty set). Further, due to Axiom 3.7 (specification), Axiom 3.5 (singletons and pairs) and Axiom 3.6 (pairwise unions) can formulated with "$\impliedby$" instead of "$\iff$".

Lemma 3.1.19 (Pairwise intersections)**.** *Let A and B be sets. Then there exists a unique set X such that for any object x, we have $x \in X \iff x \in A$ and $x \in B$.*

Remark 3.1.20. This lets us denote X by $A \cap B$. Obeys axiom of substitution.

Definition 3.1.21 (Disjoint sets)**.** Objects A and B are called disjoint sets iff A and B are sets, and $A \cap B = \emptyset$.

Lemma 3.1.22 (Difference sets)**.** *Let A and B be sets. Then there exists a unique set X such that for any object x, we have $x \in X \iff x \in A$ and $x \notin B$.*

Remark 3.1.23. This lets us denote X by $A \setminus B$. Axiom of substitution satisfied.

Proposition 3.1.24 (Sets form a Boolean algebra)**.** *Let A, B, C, X be sets. Then*

(i) (minimal element) $A \cup \emptyset = A$ *and* $A \cap \emptyset = \emptyset$,
(ii) (maximal element) $A \subseteq X \implies A \cup X = X$ *and* $A \cap X = A$,
(iii) (identity) $A \cup A = A \cap A = A$,
(iv) (commutativity) $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$,
(v) (associativity) $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$,
(vi) (distributivity) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(vii) (partition) $A \subseteq X \implies A \cup (X \setminus A) = X$ *and* $A \cap (X \setminus A) = \emptyset$, *and*
(viii) (De Morgan laws) $X \setminus (A \cup B) = (X \setminus A) \cup (X \setminus B)$ *and* $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Remark 3.1.25. (v) allows to denote $A \cup (B \cup C)$ and $(A \cup B) \cup C$ both by $A \cup B \cup C$, and $A \cap (B \cap C)$ and $(A \cap B) \cap C$ both by $A \cap B \cap C$.

Proposition 3.1.26. *Let A, B be sets. Then these are equivalent: $A \subseteq B$, $A \cup B = B$, and $A \cap B = A$.*

Proposition 3.1.27. *Let A, B, C be sets. Then $C \subseteq A, B \iff C \subseteq A \cap B$, and $A, B \subseteq C \iff A \cup B \subseteq C$.*

Proposition 3.1.28 (Absorption laws)**.** *Let A, B be sets. Then $A \cup (A \cap B) = A \cap (A \cup B) = A$.*

Proposition 3.1.29. *Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \emptyset$. Then $A = X \setminus B$ and $B = X \setminus A$.*

Proposition 3.1.30. *Let A, B be sets. Then $(A \setminus B) \cap (A \cap B) = (A \cap B) \cap (B \setminus A) = (B \setminus A) \cap (A \setminus B) = \emptyset$, and $(A \setminus B) \cup (A \cap B) \cup (B \setminus A) = A \cup B$.*

Axiom 3.8 (Replacement)**.** *Let A be a set and $P(x, y)$ be a property pertaining to any $x \in A$ and any object y such that for each $x \in A$, there is at most one object y such that $P(x, y)$ holds. Then there exists a set X such that for any object y, we have $y \in X \impliedby P(x, y)$ holds for some $x \in A$.*

Remark 3.1.31. It follows that above holds with "exists" replaced with "exists a unique", "X" replaced with "X'" and "$\impliedby$" replaced with "$\iff$". This allows to denote X' by $\{y : P(x, y) \text{ for some } x \in A\}$. Obeys axiom of substitution.

Remark 3.1.32. Axiom 3.8 (replacement) implies Axiom 3.7 (specification).

Remark 3.1.33. If $f\colon A \to B$, then we abbreviate $\{y : y = f(x) \text{ for some } x \in A\}$ by $\{f(x) : x \in A\}$.

Axiom 3.9 (Infinity)**.** *There exists a set X such that for any object x, we have $x \in X \impliedby x$ is a natural number.*

Remark 3.1.34. It follows that the above holds with "exists" replaced with "exists a unique", "X" replaced with "X'" and "$\impliedby$" replaced with "$\iff$". This allows to denote X' by $\mathbb{N}$.

3.2 Russel's Paradox

May 9, 2021

Axiom (not to be used) 3.1 (Universal specification)**.** *Let $P(x)$ be a property pertaining to any object x. Then there exists a set X such that for any object x, we have $x \in X \iff P(x)$.*

Remark 3.2.1. Existence above can be strengthened to unique existence (using only Axiom 3.3 (equality of sets)), allowing to denote X by $\{x : P(x)\}$.

Remark 3.2.2. Axiom (not to be used) 3.1 (universal specification) implies Axioms 3.4 (empty set) to 3.9 (infinity).

Remark 3.2.3. Axiom (not to be used) 3.1 and Axiom 3.3 (equality of sets) imply:

(i) Russel's paradox: There exists a unique set X such that for any object x, we have $x \in X \iff x$ is a set and $x \notin x$. Further, for any such set X', we have $X' \in X' \iff X' \notin X'$.

(ii) (Universal set)**.** There exists a unique set Y such that for any object y, we have $y \in Y \iff y$ is a an object. This allows to denote Y by Ω. We have that $\Omega \in \Omega$.

(iii) There exists a unique set Z such that for any object z, we have $z \in Z \iff z$ is a set. Further, for any such set Z', we have $Z' \in Z'$.

Remark 3.2.4. (ii) and Axiom 3.7 (specification) imply Axiom (not to be used) 3.1.

Axiom 3.10 (Regularity)**.** *Let A be a nonempty set. Then there exists an object $x \in A$ such that x is not a set, or $x \cap A = \emptyset$.*

Proposition 3.2.5. *Let A and B be sets. Then*

(i) $A \notin A$, *and*
(ii) $A \notin B$ *or* $B \notin A$.

Remark 3.2.6. This is readily generalizable to the fact that for any sets, $A_1, \ldots, A_n$, it is not the case that $A_1 \in A_2 \in \cdots \in A_n \in A_1$.

3.3 Functions

May 10, 2021

Axiom 3.11 (Functions)**.** *Functions are objects.*

Axiom 3.12 (Properties of the relation symbol $f\colon X \to Y$)**.** *(i) Let f be a function. Then there exist objects X, Y such that $f\colon X \to Y$.*

(ii) Let f, X, Y be objects such that $f\colon X \to Y$. Then

(a) f is a function and X, Y are sets,
(b) for any objects x, y, $\mathrm{OrdIn}(x, y; f) \implies x \in X$ and $y \in Y$,
(c) for each $x \in X$, there exists a unique y such that $\mathrm{OrdIn}(x, y; f)$.

(iii) Let f, X, Y, Y' be objects such that $f\colon X \to Y$ and $f\colon X \to Y'$. Then $Y = Y'$.

Axiom 3.13 (Equality of functions)**.** *Let f, g, X, Y be objects such that $f, g\colon X \to Y$ and for any objects x, y, y', let $\mathrm{OrdIn}(x, y; f)$ and $\mathrm{OrdIn}(x, y'; g) \implies y = y'$. Then $f = g$.*

Axiom 3.14 (Functions defined by functional properties)**.** *Let X, Y be sets and $P(x, y)$ be a property pertaining to any $x \in X$ and any $y \in Y$ such that for each $x \in X$, there exists a unique y such that $P(x, y)$ holds. Then there exists a function f such that $f\colon X \to Y$ and for any objects x, y, we have $\mathrm{OrdIn}(x, y; f) \implies P(x, y)$.*

Lemma 3.3.1. *Let f be a function and X, X', Y, Y' be sets such that $f\colon X \to Y$ and $f\colon X' \to Y'$. Then $X = X'$ and $Y = Y'$.*

Lemma 3.3.2 (Domains and codomains of functions)**.** *Let f be a function. Then there exist unique sets X, Y such that $f\colon X \to Y$.*

Remark 3.3.3. This allows to denote X and Y by $\operatorname{dom} f$ and $\operatorname{codom} f$ respectivelty. Axiom of substitution obeyed.

Corollary 3.3.4. *Let f be a function. Then $f\colon \operatorname{dom} f \to \operatorname{codom} f$.*

Lemma 3.3.5 (Function values at inputs)**.** *Let f be a function and $x \in \operatorname{dom} f$. Then there exists a unique y such that $\operatorname{OrdIn}(x, y; f)$.*

Remark 3.3.6. This allows to denote y by $f(x)$ or f_x. Axiom of substitution obeyed.

Lemma 3.3.7 (Equality of functions)**.** *Let f, g be functions. Then $f = g \iff \operatorname{dom} f = \operatorname{dom} g$, and $\operatorname{codom} f = \operatorname{codom} g$, and $f(x) = g(x)$ for each $x \in \operatorname{dom} f \cap \operatorname{dom} g$.*

Lemma 3.3.8 (Functions defined by functional properties)**.** *Let X, Y, be sets and $P(x, y)$ be a property pertaining to any $x \in X$ and any $y \in Y$ such that for each $x \in X$, there exists a unique y such that $P(x, y)$ holds. Then there exists a unique function f such that $f\colon X \to Y$ so that for each $x \in X$, we have $x \in \operatorname{dom} f$ and $P(x, f(x))$ holds.*

Lemma 3.3.9. *Let f be a function and X, Y be sets. Then $f\colon X \to Y \iff \operatorname{dom} f = X$ and $\operatorname{codom} f = Y$.*

Lemma 3.3.10. *Let $P(f, X, Y)$ be a property pertaining to any function f and any sets X, Y. Then the following are equivalent:*

(i) Let f be a function and X, Y be sets such that $f\colon X \to Y$. Then $P(f, X, Y)$.
(ii) Let f be a function. Then $P(f, \operatorname{dom} f, \operatorname{codom} f)$.

Lemma 3.3.11 (Function compositions)**.** *Let f, g be functions such that $\operatorname{codom} f = \operatorname{dom} g$. Then there exists a unique function h such that $h\colon \operatorname{dom} f \to \operatorname{codom} g$ and for each $x \in \operatorname{dom} f$, we have $f(x) \in \operatorname{dom} g$ and $h(x) = g(f(x))$.*

Remark 3.3.12. This allows to denote h by $g \circ f$. Axiom of substitution obeyed.

Lemma 3.3.13 (Function composition is associative)**.** *Let f, g, h be functions such that $\operatorname{codom} f = \operatorname{dom} g$ and $\operatorname{codom} g = \operatorname{dom} h$. Then $(h \circ g) \circ f = h \circ (g \circ f)$.*

Remark 3.3.14. This allows to denote $(h \circ g) \circ f$ and $h \circ (g \circ f)$ both by $h \circ g \circ f$.

Definition 3.3.15 (One-to-one functions)**.** An object f is called an injection iff f is a function and for each $x_1, x_2 \in \operatorname{dom} f$, we have $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Definition 3.3.16 (Onto functions)**.** An object f is called a surjection iff f is a function and for each $y \in \operatorname{codom} Y$, there exists an $x \in \operatorname{dom} f$ such that $y = f(x)$.

Definition 3.3.17 (Bijections)**.** An object f is called a bijection iff f is both, an injection and a surjection.

Remark 3.3.18. Axiom of substitution followed by all three above.

Proposition 3.3.19. *There exists a unique function f such that $f\colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$ and $f(n) = n + 1$ for each $n \in \mathbb{N}$. Further, such an f is bijective.*

Proposition 3.3.20. *There exists a unique function f such that $f\colon \mathbb{N} \to \mathbb{N}$ such $f(n) = n^2$ for each $n \in \mathbb{N}$. Further, such an f is injective.*

Proposition 3.3.21. *Let f, g be functions such that* $\operatorname{codom} f = \operatorname{dom} g$*. Then*

(i) f, g are both injective $\implies$ $g \circ f$ is injective,
(ii) f, g are both surjective $\implies$ $g \circ f$ is surjective,
(iii) $g \circ f$ is injective $\implies$ f is injective, and
(iv) $g \circ f$ is surjective $\implies$ g is surjective.

Proposition 3.3.22 (Cancellation law for function compositions)**.** *Let f, $\tilde{f}$, g, $\tilde{g}$ be functions such that* $\operatorname{codom} f = \operatorname{codom} \tilde{f} = \operatorname{dom} g = \operatorname{dom} \tilde{g}$*. Let f be surjective and g be injective. Then*

(i) $g \circ f = g \circ \tilde{f} \implies f = \tilde{f}$, and
(ii) $g \circ f = \tilde{g} \circ f \implies g = \tilde{g}$.

Lemma 3.3.23 (Inverses of functions)**.** *Let f be a bijection. Then there exists a unique function g such that $g\colon \operatorname{codom} f \to \operatorname{dom} f$ and for each $x \in \operatorname{dom} f$ and for each $y \in \operatorname{codom} f$, we have $g(y) = x \iff f(x) = y$.*

Remark 3.3.24. This lets us denote g by f^{-1}. Axiom of substitution holds.

Proposition 3.3.25 (Empty functions)**.** *(i) Let X be a set. Then there exists a unique function f such that $f\colon \emptyset \to X$.*
(ii) Let f be a function such that $\operatorname{dom} f = \emptyset$*. Then*

(a) f is injective, and
(b) f is surjective $\iff$ $\operatorname{codom} f = \emptyset$.

Proposition 3.3.26. *Let f be a bijection. Then for each $x \in \operatorname{dom} f$, we have $f^{-1}(f(x)) = x$ and for each $y \in \operatorname{codom} f$, we have $f(f^{-1}(y)) = y$. Further, f^{-1} is also a bijection with $(f^{-1})^{-1} = f$.*

Proposition 3.3.27 (Inverses of compositions)**.** *Let f, g be bijections such that $\operatorname{codom} f = \operatorname{dom} g$. Then $g \circ f$ is a bijection. Further, $\operatorname{codom} g^{-1} = \operatorname{dom} f^{-1}$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Lemma 3.3.28 (Inclusion maps)**.** *Let X, Y be sets such that $X \subseteq Y$. Then there exists a unique function f such that $f\colon X \to Y$ and for each $x \in X$, $f(x) = x$.*

Remark 3.3.29. This lets us denote f by $\iota_{X \to Y}$. Axiom of substitution obeyed.

Proposition 3.3.30. *Let X, Y, Z be sets and f, g be functions. Then*

(i) $X \subseteq Y \subseteq Z \implies \iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$,
(ii) $f\colon X \to Y \implies f = f \circ \iota_{X \to X} = \iota_{Y \to Y} \circ f$,
(iii) $f\colon X \to Y$ and f is invertible $\implies f^{-1} \circ f = \iota_{X \to X}$ and $f \circ f^{-1} = \iota_{Y \to Y}$,
(iv) if $f\colon X \to Z$ and $g\colon Y \to Z$ such that for each $x \in X \cap Y$, we have $f(x) = g(x)$, then there exists a unique function h such that $h\colon X \cup Y \to Z$, $h \circ \iota_{X \to X \cup Y} = f$ and $h \circ \iota_{Y \to X \cup Y} = g$. Further, for such a function h, we have $h(x) = f(x)$ for any $x \in X$ and $h(y) = g(y)$ for any $y \in Y$.

3.4 Images and inverse images

May 11, 2021

Lemma 3.4.1 (Forward images of sets)**.** *Let f be a function and S be a set such that $S \subseteq \operatorname{dom} f$. Then there exists a unique set Z such that for any object y, we have $y \in Z \iff y = f(x)$ for some $x \in S$.*

Remark 3.4.2. This lets us denote Z by $f[S]$. Axiom of substitution obeyed.

Corollary 3.4.3. *Let f be a function. Then f is surjective $\iff$ $f[\operatorname{dom} f] = \operatorname{codom} f$.*

Lemma 3.4.4 (Inverse images of sets)**.** *Let f be a function and U be a set such that $U \subseteq \operatorname{codom} f$. Then there exists a unique set Z such that for any object x, we have $x \in Z \iff x \in \operatorname{dom} f$ and $f(x) \in U$.*

Lemma 3.4.5. *Let f be a bijection and S be a set such that $S \subseteq \operatorname{dom} f^{-1}$. Then $S \subseteq \operatorname{codom} f$. Let Z be a set such that for any object x, we have $x \in Z \iff x \in \operatorname{dom} f$ and $f(x) \in S$. Then $f^{-1}[S] = Z$.*

Remark 3.4.6. Lemmas 3.4.4 and 3.4.5 let us denote Z of Lemma 3.4.4 by $f^{-1}[U]$. Axiom of substitution obeyed.

Proposition 3.4.7 (Forward images of inverse images and vice versa)**.** *Let f be a function and S, U be sets such that $S \subseteq \operatorname{dom} f$ and $U \subseteq \operatorname{codom} f$. Then*

(i) $f[S] \subseteq \operatorname{codom} f$ and $f^{-1}[f[S]] \supseteq S$, and
(ii) $f^{-1}[U] \subseteq \operatorname{dom} f$ and $f[f^{-1}[U]] \subseteq U$.

Proposition 3.4.8 (Properties of forward images)**.** *Let f be a function and A, B be sets such that $A, B \subseteq \operatorname{dom} f$. Then*

(i) $f[A \cup B] = f[A] \cup f[B]$,
(ii) $f[A \cap B] \subseteq f[A] \cap f[B]$, and
(iii) $f[A \setminus B] \supseteq f[A] \setminus f[B]$.

Proposition 3.4.9 (Properties of inverse images)**.** *Let f be a function and A, B be sets such that $A, B \subseteq \operatorname{codom} f$. Then*

(i) $f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$,
(ii) $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$, and
(iii) $f^{-1}[A \setminus B] = f^{-1}[A] \setminus f^{-1}[B]$.

Proposition 3.4.10 (Forward and inverse images of compositions)**.** *Let f, g be functions such that $\operatorname{codom} f = \operatorname{dom} g$. Let A, B be sets such that $A \subseteq \operatorname{dom} f$ and $B \subseteq \operatorname{codom} g$. Then*

(i) $(g \circ f)[A] = g[f[A]]$, and
(ii) $(g \circ f)^{-1}[B] = f^{-1}[g^{-1}[B]]$.

Proposition 3.4.11. *Let f be a function. Then*

(i) f is injective $\iff$ for each set S, we have $S \subseteq \operatorname{dom} f \implies f^{-1}[f[S]] = S$, and
(ii) f is surjective $\iff$ for each set U, we have $U \subseteq \operatorname{codom} f \implies f[f^{-1}[U]] = U$.

Axiom 3.15 (Sets of functions for given domains and codomains)**.** *Let X, Y be sets. Then there exists a set Z such that for any object f, we have $f \in Z \impliedby f\colon X \to Y$.*

Remark 3.4.12. It follows that above holds with "exists" replaced with "exists a unique", "Z" replaced with "Z'" and " $\impliedby$ " replaced with " $\iff$ ". This allows to denote Z' by Y^X. Axiom of substitution obeyed.

Lemma 3.4.13 (Power sets)**.** *Let X be a set. Then there exists a unique set Z such that for any object Y, we have $Y \in Z \iff Y \subseteq X$.*

Remark 3.4.14. This allows to denote Z by 2^X. Axiom of substitution obeyed.

Lemma 3.4.15. *Let X be a set. Then there exists a bijection f such that $f\colon \{0,1\}^X \to 2^X$.*

Definition 3.4.16. For an object A, we write

(i) "A is a set of sets" iff A is a set and for each object X, we have that $X \in A \implies X$ is a set, and
(ii) "A is a nonempty set of sets" iff A is a set of sets and $A \neq \emptyset$.

Remark 3.4.17. Axiom of substitution satisfied.

Axiom 3.16 (Unions)**.** *Let A be a set of sets. Then there exists a set Z such that for any object x, we have $x \in Z \impliedby x \in X$ for some set $X \in A$.*

Remark 3.4.18. It follows that above holds with "exists" replaced with "exists a unique", "Z" replaced with "Z'" and " $\impliedby$ " replaced with " $\iff$ ". This allows to denote Z' by $\bigcup A$. Axiom of substitution obeyed.

Remark 3.4.19. Axiom 3.16 (unions) and Axiom 3.5 (ii) (pair sets) along with Axiom 3.1 (sets) imply Axiom 3.6 (pairwise unions).

Corollary 3.4.20. $\bigcup \emptyset = \emptyset$.

Lemma 3.4.21 (Intersections)**.** *Let A be a nonempty set of sets. Then there exists a unique set Z such that for any object x, we have $x \in Z \iff x \in X$ for each set $X \in A$.*

Remark 3.4.22. This allows to denote Z by $\bigcap A$. Axiom of substitution obeyed.

Definition 3.4.23 (Partial functions)**.** For objects f, X, Y, we write "f is a partial function from X to Y" iff there exist objects X', Y' such that $X' \subseteq X$, $Y' \subseteq Y$ and $f\colon X' \to Y'$.

Remark 3.4.24. Axiom of substitution obeyed.

Proposition 3.4.25 (Sets of partial functions)**.** *Let X, Y be sets. Then there exists a unique set Z such that for any object f, we have $f \in Z \iff$ f is a partial function from X to Y.*

Definition 3.4.26 (Families)**.** For objects X, I, we write

(i) "$\{X_\alpha\}_{\alpha\in I}$ is a family", or "X is a surjection on domain I" iff X is a surjection and $I = \operatorname{dom} X$,
(ii) "$\{X_\alpha\}_{\alpha\in I}$ is a family of sets" iff $\{X_\alpha\}_{\alpha\in I}$ is a family and for each $\alpha \in I$, X_α is a set, and
(iii) "$\{X_\alpha\}_{\alpha\in I}$ is a nonempty family of sets" iff $\{X_\alpha\}_{\alpha\in I}$ is a family of sets and I is nonempty.

Remark 3.4.27. Axiom of substitution obeyed.

Remark 3.4.28. We are not positing families to be a distinct object type and their equality need thus not be defined.

Lemma 3.4.29 (Unions and intersections of families of sets)**.** *Let A be a surjection and J be a set such that $\{A_\alpha\}_{\alpha\in\operatorname{dom} A}$ is a family of sets and $J \subseteq \operatorname{dom} A$. Then there exists a set unique Z such that for any object x, we have $x \in Z \iff x \in A_\alpha$ for some $\alpha \in J$. Furthermore, if J is nonempty, then there exists a unique set Z' such that for any object x, we have $x \in Z' \iff$ $x \in A_\alpha$ for each $\alpha \in J$.*

Remark 3.4.30. This allows to denote Z and Z' by $\bigcup_{\alpha\in J} A_\alpha$ and $\bigcap_{\alpha\in J} A_\alpha$ respectively. Axiom of substitution obeyed.

Proposition 3.4.31. *Let A be a surjection and I, J be sets such that $\{A_\alpha\}_{\alpha\in I\cup J}$ is a family of sets. Then*

(i) $\left(\bigcup_{\alpha\in I} A_\alpha\right) \cup \left(\bigcup_{\alpha\in J} A_\alpha\right) = \bigcup_{\alpha\in I\cup J} A_\alpha$, *and*
(ii) If I, J are nonempty, then $I\cup J$ is nonempty and $\left(\bigcap_{\alpha\in I} A_\alpha\right)\cap\left(\bigcap_{\alpha\in J} A_\alpha\right) = \bigcap_{\alpha\in I\cup J} A_\alpha$.

Lemma 3.4.32. *Let A be a surjection and J, X be sets such that $\{A_\alpha\}_{\alpha \in \operatorname{dom} A}$ is a family of sets and $J \subseteq \operatorname{dom} A$. Then there exists a unique set Z such that for each object z, we have $z \in Z \iff z \in X \setminus A_\alpha$ for some $\alpha \in J$. Further, if J is nonempty, then there exists a set Z' such that for any object z, we have $z \in Z' \iff z \in X \setminus A_\alpha$ for each $\alpha \in J$.*

Remark 3.4.33. This lets us denote Z and Z' by $\bigcup_{\alpha \in J}(X \setminus A_\alpha)$ and $\bigcap_{\alpha \in J}(X \setminus A_\alpha)$ respectively. Axiom of substitution obeyed.

Proposition 3.4.34 (De Morgan laws for families of sets)**.** *Let $\{A_\alpha\}_{\alpha \in I}$ be a nonempty family of sets and X be a set. Then*

(i) $X \setminus \left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcap_{\alpha \in I}(X \setminus A_\alpha)$, *and*
(ii) $X \setminus \left(\bigcap_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I}(X \setminus A_\alpha)$.

3.5 Cartesian products

May 20, 2021

Axiom 3.17 (Ordered pairs)**.** *Ordered pairs are objects.*

Axiom 3.18 (Properties of the function symbol (x, y))**.** *(i) Let x, y be objects. Then (x, y) is an ordered pair.*
(ii) Let x, y, x', y' be objects such that $(x, y) = (x', y')$. Then $x = x'$ and $y = y'$.
(iii) Let p be an ordered pair. Then there exist objects x, y such that $p = (x, y)$.

Remark 3.5.1. The domain of the function symbol "(,)" is all objects in both slots.

Remark 3.5.2. Consistency of equality for ordered pairs as an equivalence relation depends on the equality of objects being consistent as equivalent relations.

Lemma 3.5.3 (Components of ordered pairs)**.** *Let p be an ordered pair. Then there exist unique objects x, y such that $p = (x, y)$.*

Remark 3.5.4. This allows to denote x and y by p_1 and p_2 respectively. Axiom of substitution obeyed.

Corollary 3.5.5. *Let p be an ordered pair. Then $p = (p_1, p_2)$.*

Lemma 3.5.6. *Let p be an ordered pair and x, y be objects. Then $p = (x, y) \iff x = p_1$ and $y = p_2$.*

Lemma 3.5.7 (Equality of ordered pairs)**.** *Let p, q be ordered pairs. Then $p = q \iff p_1 = q_1$ and $p_2 = q_2$.*

Lemma 3.5.8. *Let $P(p, x, y)$ be a property pertaining to any ordered pair p, and any objects x, y. Then the following are equivalent:*

(i) Let p be an ordered pair and x, y by objects such that $p = (x, y)$. Then $P(p, x, y)$.
(ii) Let p be an ordered pair. Then $P(p, p_1, p_2)$.

Proposition 3.5.9 (Pairwise Cartesian products)**.** *Let X, Y be sets. Then there exists a unique set Z such that for any object p, we have $p \in Z \iff p$ is an ordered pair with $p_1 \in X$ and $p_2 \in Y$.*

Remark 3.5.10. This allows to denote Z by $X \times Y$. Axiom of substitution obeyed.

Lemma 3.5.11. *Let X, Y be sets and x, y be objects. Then $(x, y) \in X \times Y \iff x \in X$ and $y \in Y$.*

Alternate definition 3.5.12 (Making functions, ordered pairs)**.** (i) For objects x, y, f, we write "OrdIn$(x, y; f)$" iff f is an ordered pair and $(x, y) \in f_2$.

(ii) For objects f, X, Y, we write $f\colon X \to Y$ iff
   (a) X, Y are sets and f is an ordered pair,
   (b) $f_1 = Y$ and $f_2 \subseteq X \times Y$, and
   (c) for each $x \in X$, there exists a unique y such that OrdIn$(x, y; f)$.

(iii) An object f is called a function iff there exist objects X, Y such that $f\colon X \to Y$.

Remark 3.5.13. Axiom of substitution obeyed by all.

Remark 3.5.14. This does the following:

(i) Functions now become ordered pairs and equality of functions becomes equality of ordered pairs.
(ii) Axioms 3.11 to 3.14 become theorems.

Alternate definition 3.5.15 (Making ordered pairs, sets)**.** (i) For objects x, y, we set $(x,y) \coloneqq \{\{x\},\{x,y\}\}$. Alternatively, we could also set $(x,y) \coloneqq \{x,\{x,y\}\}$.

(ii) An object p is called an ordered pair iff there exist objects x, y such that $p=(x,y)$.

Remark 3.5.16. Axiom of substitution obeyed by both.

Remark 3.5.17. This does the following:

(i) Ordered pairs now become sets, and their equality becomes set equality.
(ii) Axioms 3.17 and 3.18 become theorems.

Proposition 3.5.18 (Pairwise Cartesian product distributes over set operations)**.** *Let A, B, C be sets. Then*

(i) $A\times(B\cup C)=(A\times B)\cup(A\times C)$ *and* $(A\cup B)\times C=(A\times C)\cup(B\times C)$,
(ii) $A\times(B\cap C)=(A\times B)\cap(A\times C)$ *and* $(A\cap B)\times C=(A\times C)\cap(B\times C)$, *and*
(iii) $A\times(B\setminus C)=(A\times B)\setminus(A\times C)$ *and* $(A\setminus B)\times C=(A\times C)\setminus(B\times C)$.

Proposition 3.5.19. *Let A, B, C, D be sets. Then*

(i) $(A\times B)\cup(C\times D)\subseteq(A\cup C)\times(B\cup C)$,
(ii) $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$, *and*
(iii) $(A\times B)\setminus(C\times D)\supseteq(A\setminus C)\times(B\setminus D)$.

Proposition 3.5.20. *Let A, B, C, D be sets. Then*

(i) $A\subseteq C$ *and* $B\subseteq D \implies A\times B\subseteq C\times D$.
(ii) $A\times B\neq\emptyset$ *and* $A\times B\subseteq C\times D \implies A\subseteq C$ *and* $B\subseteq D$, *and*
(iii) $A\times B\neq\emptyset$ *and* $A\times B=C\times D \implies A=C$ *and* $B=D$.

Lemma 3.5.21 (Coordinate functions)**.** *Let X, Y be sets. Then there exist unique functions f, g such that $f\colon X\times Y\to X$ and $g\colon X\times Y\to Y$ so that $f((x,y))=x$ and $g((x,y))=y$ for each $x\in X$ and for each $y\in Y$. Further, for any such functions f, g, we have $(f(p),g(p))=p$ for any $p\in X\times Y$*

Remark 3.5.22. This allows to denote f and g by $\pi_{X\times Y\to X}$ and $\pi_{X\times Y\to Y}$ respectively. Axiom of substitution obeyed.

Lemma 3.5.23 (Direct sums of functions)**.** *Let f, g be functions such that $\operatorname{dom} f=\operatorname{dom} g$. Then there exists a unique function h such that $h\colon \operatorname{dom} f\cap\operatorname{dom} g\to\operatorname{codom} f\times\operatorname{codom} g$ and $\pi_{\operatorname{codom} f\times\operatorname{codom} g\to\operatorname{codom} f}\circ h=f$ and $\pi_{\operatorname{codom} f\times\operatorname{codom} g\to\operatorname{codom} g}\circ h=g$. Further, for any such function h, we have $h(z)=(f(z),g(z))$ for any $z\in\operatorname{dom} f\cap\operatorname{dom} g$.*

Remark 3.5.24. This lets us denote h by $f \oplus g$. Axiom of substitution obeyed.

Lemma 3.5.25. *Let I, J be sets and A, B be surjections such that $I \subseteq \operatorname{dom} A$, and $J \subseteq \operatorname{dom} B$, and $\{A_i\}_{i \in \operatorname{dom} A}$ and $\{B_i\}_{i \in \operatorname{dom} B}$ are families of sets. Then there exists a unique set Z such that for any object z, we have $z \in Z \iff z \in A_\alpha \cap B_\beta$ for some $\alpha \in A$ and for some $\beta \in B$.*

Remark 3.5.26. This allows to denote Z by $\bigcup_{(\alpha,\beta) \in I \times J}(A_\alpha \cap B_\beta)$. Axiom of substitution obeyed.

Proposition 3.5.27. *Let I, J be sets and A, B be surjections such that $I \subseteq \operatorname{dom} A$, and $J \subseteq \operatorname{dom} B$, and $\{A_i\}_{i \in \operatorname{dom} A}$ and $\{B_i\}_{i \in \operatorname{dom} B}$ are families of sets. Then $\left(\bigcup_{\alpha \in I} A_\alpha\right) \cap \left(\bigcup_{\beta \in J} B_\beta\right) = \bigcup_{(\alpha,\beta) \in I \times J}(A_\alpha \cap B_\beta)$.*

Lemma 3.5.28 (Graph of a function)**.** *Let f be a function. Then there exists a unique set Z such that for any object p, we have $p \in Z \iff p = (x, f(x))$ for some $x \in \operatorname{dom} f$.*

Remark 3.5.29. This allows to denote Z be $\operatorname{graph} f$. Axiom of substitution satisfied.

Lemma 3.5.30 (Properties of graphs of functions)**.** *(i) Let f, g be functions such that $\operatorname{graph} f = \operatorname{graph} g$. Then $\operatorname{dom} f = \operatorname{dom} g$.*

(ii) Let f be a function. Then $\operatorname{graph} f \subseteq \operatorname{dom} f \times \operatorname{codom} f$ and $\operatorname{dom} f = \pi_{\operatorname{dom} f \times \operatorname{codom} f \to \operatorname{dom} f}[\operatorname{graph} f]$.

Proposition 3.5.31 (Equality of functions via graphs)**.** *Let f, g be functions. Then $f = g \iff \operatorname{codom} f = \operatorname{codom} g$ and $\operatorname{graph} f = \operatorname{graph} g$.*

Proposition 3.5.32 (Determining functions from graphs and codomains)**.** *Let X, Y, G be sets such that $G \subseteq X \times Y$ and for each $x \in X$ there exists a unique object y such that $(x, y) \in G$. Then there exists a unique function f such that $\operatorname{codom} f = Y$ and $\operatorname{graph} f = G$. Further, for such an f, we have $\operatorname{dom} f = X$.*

Remark 3.5.33. Lemma 3.4.13 (power sets) and other axioms imply Axiom 3.15 (sets of functions for given domains and codomains).

Remark 3.5.34. The following, from Definition 3.5.35 to Proposition 3.5.46, except Corollary 3.5.43 which isn't used elsewhere in the mentioned part, requires only Axioms 2.1 and 2.2, and the set theory developed so far. So no circularity.

Definition 3.5.35 (Cutting $\mathbb{N}$ at some N)**.** For objects N, A, B, we'll write "N cuts $\mathbb{N}$ into A and B" iff $N \in \mathbb{N}$ and A, B are sets, and the following hold:

(i) $A \cup B = \mathbb{N}$.
(ii) $A \cap B = \emptyset$.
(iii) $0 \in A$.
(iv) $N{+}{+} \in B$.
(v) For every $n \in \mathbb{N}$, we have $n \in B \implies n{+}{+} \in B$.
(vi) For every $n \in \mathbb{N}$, we have $n \in A \setminus \{N\} \implies n \in A$.

Remark 3.5.36. Axiom of substitution satisfied.

Remark 3.5.37. (i)-(vi) are independent for each $N \in \mathbb{N} \setminus \{0\}$. For $N = 0$, (vi) is implied by the remaining (i)-(v), which are still, however, independent.

Lemma 3.5.38. *Let N cut $\mathbb{N}$ into A and B. Then $N \notin A \implies n \in A$ and $n \neq N$ for each $n \in \mathbb{N}$.*

Corollary 3.5.39. *Let N cut $\mathbb{N}$ into A and B. Then $N \in A$.*

Proposition 3.5.40 (Uniqueness of the N-cut on $\mathbb{N}$)**.** *Let $N \in \mathbb{N}$. Then there exist unique sets A, B such that N cuts $\mathbb{N}$ into A and B.*

Remark 3.5.41. This allows to denote A by $\{0, \ldots, N\}$. (We have B then given by $\mathbb{N} \setminus \{0, \ldots, N\}$.) Axiom of substitution obeyed.

Lemma 3.5.42. *(i) $\{0, \ldots, 0\} = \{0\}$.*
(ii) Let $N \in \mathbb{N}$. Then $\{0, \ldots, N{+}{+}\} = \{0, \ldots, N\} \cup \{N{+}{+}\}$.

Corollary 3.5.43. *Let $n \in \mathbb{N}$. Then $\{1, \ldots, n\} = \{i \in \mathbb{N} : i \leq n\}$.*

Lemma 3.5.44. *Let X be a set, $c \in X$ and f be a function such that $f\colon \mathbb{N} \times X \to X$. Let $N \in \mathbb{N}$. Then there exists a unique function b such that $b\colon \{0, \ldots, N\} \to X$ so that $b(0) = c$ and for each $n \in \{0, \ldots, N\} \setminus \{N\}$, we have $b(n{+}{+}) = f((n, b(n)))$.*

Proposition 3.5.45 (Recursive definitions, rigorously)**.** *Let X be a set, $c \in X$ and f be a function such that $f\colon \mathbb{N} \times X \to X$. Then there exists a unique function a such that $a\colon \mathbb{N} \to X$ so that $a(0) = c$ and for each $n \in \mathbb{N}$, we have $a(n{+}{+}) = f((n, a(n)))$.*

Proposition 3.5.46 (Peano axioms are categorical)**.** *Let Axioms 2.1 (natural numbers), 2.2 (Peano axioms) and 3.9 (infinity) hold for another type of objects, with* 0 *replaced with* $0'$ *and* $++$ *replaced with* $++'$*, and let the set containing all and only objects of this new type be denoted* $\mathbb{N}'$*. Then there exists a unique function* f *such that* $f\colon \mathbb{N} \to \mathbb{N}'$ *such that* $f(0) = 0'$ *and* $f(n{++}) = f(n){++}$ *for each* $n \in \mathbb{N}$*. Further, any such* f *is a bijection.*

Remark 3.5.47. Now, we can freely use all the properties of $\mathbb{N}$ developed so far.

Remark 3.5.48. We write "let $x \in A$" to abbreviate "let x be an object and A be a set such that $x \in A$".

Lemma 3.5.49 (Initial segments of naturals)**.** *Let* $n \in \mathbb{N}$*. Then there exists a unique set* X *such that for any object* m*, we have* $m \in X \iff m \in \mathbb{N}$ *and* $1 \le m \le n$*.*

Remark 3.5.50. This lets us denote X by $\{1, \dots n\}$. Axiom of substitution obeyed.

Lemma 3.5.51. *(i)* $\{1, \dots, 0\} = \emptyset$*.*
(ii) Let $n \in \mathbb{N}$*. Then* $\{1, \dots, n+1\} = \{1, \dots, n\} \cup \{n+1\}$*.*

Lemma 3.5.52. *Let* $m, n \in \mathbb{N}$ *such that* $\{1, \dots, m\} = \{1, \dots, n\}$*. Then* $m = n$*.*

Definition 3.5.53 (Ordered n-tuples)**.** For objects n, X, we write "X is an ordered n-tuple" iff $n \in \mathbb{N}$ and X is a surjection on domain $\{1, \dots, n\}$.

Remark 3.5.54. Axiom of substitution obeyed.

Proposition 3.5.55 (Empty tuple)**.** *There exists a unique object* P *such that* P *is an ordered* 0*-tuple.*

Remark 3.5.56. This allows to denote P by $()$.

Proposition 3.5.57 (n-fold Cartesian products)**.** *Let* $n \in \mathbb{N}$ *and* X *be a surjection on domain* $\{1, \dots, n\}$ *such that* X_i *is a set for each* $1 \le i \le n$*. Then there exists a unique set* Z *such that for any object* P*, we have* $P \in Z \iff P$ *is an ordered* n*-tuple such that* $P_i \in X_i$ *for each* $1 \le i \le n$*.*

Remark 3.5.58. This allows to denote Z by $\prod_{i=1}^{n} X_i$. Axiom of substitution obeyed.

Proposition 3.5.59. *Let X be a surjection on domain $\{1, \dots, 0\}$. Then X_i is a set for each $1 \leq i \leq 0$ and $\prod_{i=1}^{0} X_i = \{()\}$.*

Lemma 3.5.60. *Let A be a set and x be an object. Then there exists a unique function f such that $f \colon A \to \{x\}$.*

Proposition 3.5.61. *Let $n \in \mathbb{N}$ and X be a set. Then there exists a unique set Z such that for any object P, we have $P \in Z \iff P$ is an ordered n-tuple such that $P_i \in X$ for each $1 \leq i \leq n$.*

Remark 3.5.62. This allows to denote Z by $\prod_{i=1}^{n} X$, or X^n. Axiom of substitution obeyed.

Proposition 3.5.63. *Let X be a set. Then $X^0 = \{()\}$.*

May 27, 2021

Lemma 3.5.64 (Finite choice)**.** *Let $n \in \mathbb{N}$ and X be a surjection on domain $\{1, \dots, n\}$ such that $n \geq 1$ and X_i is a nonempty set for each $1 \leq i \leq n$. Then $\prod_{i=1}^{n} X_i$ is nonempty.*

Proposition 3.5.65. *Let $n \in \mathbb{N}$ and X be a surjection on domain $\{1, \dots, n\}$ such that X_i is a set for each $1 \leq i \leq n$. Then $\prod_{i=1}^{n} X_i = \emptyset \iff X_i = \emptyset$ for some $1 \leq i \leq n$.*

Proposition 3.5.66 (Generalized recursive definitions)**.** *Let X be a set, and $c \in X$ and $\{g_n\}_{n \in \mathbb{N}}$ be a family such that g_n is a function for each $n \in \mathbb{N}$ such that $g_n \colon X^{n+1} \to X$. Then there exists a unique function $h \colon \mathbb{N} \to X$ such that $h(0) = c$ and for each $n \in \mathbb{N}$, and for any function P such that P is an ordered $(n+1)$-tuple P so that $P_{i+1} = h(i)$ for each $0 \leq i \leq n$, we have $h(n+1) = g_n(P)$.*

Further, for any $n \in \mathbb{N}$, such a function P is unique.

Remark 3.5.67. The primitive object types so far:

(i) Natural numbers
(ii) Sets
(iii) Functions
(iv) Ordered pairs

Remark 3.5.68. Primitive relations and function symbols and constants so far:

(i) 0, $++$, equality for naturals
(ii) $\in$, equality for sets
(iii) $f\colon X \to Y$, OrdIn, equality for functions
(iv) (x, y), equality for ordered pairs

Remark 3.5.69. Using Alternate definitions 3.5.12 and 3.5.15, we can eliminate the last two items in both lists above.

3.6 Cardinality of sets

June 2, 2021

Definition 3.6.1 (Equal cardinalities)**.** For objects X, Y, we write "X and Y have equal cardinality" iff X, Y are sets and there exists a bijection f such that $f\colon X \to Y$.

Remark 3.6.2. Axiom of substitution obeyed.

Proposition 3.6.3 ($\mathbb{N}$, odds, and evens are equinumerous)**.** *(i)* $\mathbb{N}$ *and* $\{2n : n \in \mathbb{N}\}$ *have equal cardinality.*
(ii) $\mathbb{N}$ *and* $\{2n + 1 : n \in \mathbb{N}\}$ *have equal cardinality.*

Proposition 3.6.4 ("Equal cardinality" is an equivalence relation)**.** *Let X, Y, Z be sets. Then*

(i) (reflexivity) *X and X have equal cardinality,*
(ii) (symmetry) *X and Y have equal cardinality $\implies$ Y and X have equal cardinality, and*
(iii) (transitivity) *X and Y have equal cardinality, and Y and Z have equal cardinality $\implies$ X and Z have equal cardinality.*

Definition 3.6.5 (Sets having n elements)**.** For objects X, n, we write "X has n elements" or "X has cardinality n" iff $n \in \mathbb{N}$, and X and $\{1, \dots, n\}$ have equal cardinality.

Definition 3.6.6 (Finite sets)**.** An object X is called a finite set iff there exists an object n such that X has n elements.

Remark 3.6.7. Axiom of substitution obeyed by both.

Lemma 3.6.8. *Let X be a set. Then $X = \emptyset \iff X$ has 0 elements.*

Lemma 3.6.9. *Let $n \in \mathbb{N}$. Then $\{i \in \mathbb{N} : i < n\}$ has n elements.*

Lemma 3.6.10. *(i) $\{i \in \mathbb{N} : i < 0\} = \emptyset$.*
(ii) Let $n \in \mathbb{N}$. Then $\{i \in \mathbb{N} : i < n + 1\} = \{i \in \mathbb{N} : i < n\} \cup \{n\}$.

Lemma 3.6.11. *Let X, Y be sets having equal cardinality. Let $x_0 \in X$ and $y_0 \in Y$. Then $X \setminus \{x_0\}$ and $Y \setminus \{y_0\}$ have equal cardinality.*

Proposition 3.6.12. *Let $m, n \in \mathbb{N}$, and $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ have equal cardinality. Then $m = n$.*

Corollary 3.6.13. *Let $n \in \mathbb{N}$, X be a set having $n+1$ elements and $x_0 \in X$. Then $X \setminus \{x_0\}$ has n elements.*

Corollary 3.6.14 (Cardinality of finite sets)**.** *Let X be a finite set. Then there exists a unique $n \in \mathbb{N}$ such that X has n elements.*

Remark 3.6.15. This allows to denote n by $\#(X)$. Axiom of substitution obeyed.

Definition 3.6.16. An object X is called an infinite set iff X is a set and X is not a finite set.

Remark 3.6.17. Axiom of substitution obeyed.

Lemma 3.6.18 (Finite sequences in $\mathbb{N}$ are bounded)**.** *Let $n \in \mathbb{N}$ and f be a function such that $f\colon \{1, \ldots, n\} \to \mathbb{N}$. Then there exists an $M \in \mathbb{N}$ such that $f(i) \leq M$ for each $1 \leq i \leq n$.*

Corollary 3.6.19 (Finite subsets of $\mathbb{N}$ are bounded)**.** *Let X be a finite set such that $X \subseteq \mathbb{N}$. Then there exists an $M \in \mathbb{N}$ such that $M \geq i$ for each $i \in X$.*

Theorem 3.6.20. $\mathbb{N}$ *is an infinite set.*

Lemma 3.6.21. *Let $P(n, Y)$ be a property pertaining to any $n \in \mathbb{N}$ and any set Y. Then the following are equivalent:*
(i) Let $n \in \mathbb{N}$ and X be a set having n elements. Then $P(n, X)$.
(ii) Let X be a finite set. Then $P(\#(X), X)$.

Lemma 3.6.22. *Let X, Y be sets with equal cardinality such that X is finite. Then Y is also finite and $\#(X) = \#(Y)$.*

Proposition 3.6.23 (Properties of cardinalities of finite sets)**.** *(i) Let X be a finite set and $x_0 \notin X$. Then $X \cup \{x_0\}$ is a finite set and $\#(X \cup \{x_0\}) = \#(X) + 1$.*

(ii) Let X, Y be finite sets. Then $X \cup Y$ and $X \cap Y$ are finite sets and $\#(X \cup Y) + \#(X \cap Y) = \#(X) + \#(Y)$.

(iii) Let X be a finite set and $Y \subsetneq X$. Then Y is finite and $\#(Y) < \#(X)$.

Lemma 3.6.24. *Let X be a set and f be a function such that* $\operatorname{dom} f = X$ *and f is not injective. Let $x_0 \in X$ and $f \circ \iota_{X \setminus \{x_0\} \to X}$ be injective. Then $f(x_0) \in f[X \setminus \{x_0\}]$.*

Proposition 3.6.25 (Cardinalities of images of finite domains)**.** *Let X be a finite set and f be a function such that* $\operatorname{dom} f = X$*. Then $f[X]$ is a finite set and*

(i) f is an injection $\implies \#(f[X]) = \#(X)$, and

(ii) f is not an injection $\implies \#(f[X]) < \#(X)$.

Lemma 3.6.26. *Let X be a set and a be an object. Then X and $\{a\} \times X$ have equal cardinality.*

Proposition 3.6.27 (Cardinalities of Cartesian products of finite sets)**.** *Let X, Y be finite sets. Then $X \times Y$ is a finite set with $\#(X \times Y) = \#(X)\#(Y)$.*

Lemma 3.6.28 (Cardinality of singletons)**.** *Let a be an object. Then $\{a\}$ is a finite set with $\#(\{a\}) = 1$.*

Lemma 3.6.29. *Let X, Y be sets and $x_0 \in X$. Then Y^X and $Y^{X \setminus \{x_0\}} \times Y$ have equal cardinality.*

Proposition 3.6.30 (Cardinalities of function sets for finite domains and codomains)**.** *Let X, Y be finite sets. Then Y^X is a finite set with $\#(Y^X) = \#(Y)^{\#(X)}$.*

Proposition 3.6.31. *Let A, B, C be sets. Then*

(i) $A \times B$ and $B \times A$ have equal cardinality,

(ii) $(C^B)^A$ and $C^{B \times A}$ have equal cardinality, and

(iii) $A \cap B = \emptyset \implies C^B \times C^A$ and $C^{B \cup A}$ have equal cardinality.

Lemma 3.6.32. *Let $m, n \in \mathbb{N}$. Set $X := \{i \in \mathbb{N} : m + 1 \leq i \leq m + n\}$. Then*

(i) *X has n elements, and*
(ii) $\{1, \ldots, m\} \cap X = \emptyset$.

Remark 3.6.33. One can prove the obviously corresponding properties of naturals using the above.

Definition 3.6.34 (Less or equal cardinalities)**.** For objects X and Y, we write "X has cardinality less than or equal to that of Y" iff there exists an injection f such that $f\colon X \to Y$.

Remark 3.6.35. Axiom of substitution obeyed.

Proposition 3.6.36. *Let X, Y be finite sets. Then X has cardinality less than or equal to that of $Y \iff \#(X) \leq \#(Y)$.*

Proposition 3.6.37. *Let A, B be sets such that $A \neq \emptyset$ and A has cardinality less than or equal to that of B. Then there exists a surjection g such that $g\colon B \to A$.*

Remark 3.6.38. If $m, n \in \mathbb{N}$, and $\{A_i\}_{i \in \{1,\ldots,n\}}$ is a family of sets and $m \leq n$, then we set $\bigcup_{i=1}^{m} A_i \coloneqq \bigcup_{i \in \{1,\ldots,m\}} A_i$. Further, if $m \geq 1$, then we set we set $\bigcap_{i=1}^{m} A_i \coloneqq \bigcap_{i \in \{1,\ldots,m\}} A_i$.

Proposition 3.6.39 (Pigeonhole principle)**.** *Let $n \in \mathbb{N}$ and A be a surjection on domain $\{1, \ldots, n\}$ such that A_i is a finite set for each $1 \leq i \leq n$. Then*
(i) *$\bigcup_{i=1}^{n} A_i$ is finite, and*
(ii) $\#(\bigcup_{i=1}^{n} A_i) > n \implies \#(A_i) \geq 2$ *for some* $1 \leq i \leq n$.

Chapter 4

Integers and rationals

4.1 The integers

June 3, 2021

Axiom 4.1 (Integers). *Integers are objects.*

Axiom 4.2 (Properties of the function symbol $a —— b$). *(i) Let $a, b \in \mathbb{N}$. Then $a —— b$ is an integer.*

(ii) Let $a, b, c, d \in \mathbb{N}$. Then $a —— b = c —— d \iff a + d = c + b$.

(iii) Let p be an integer. Then there exist $a, b \in \mathbb{N}$ such that $p = a —— b$.

Remark 4.1.1. The domain of the function symbol " —— " is naturals in both slots.

Lemma 4.1.2 (Consistency of equality of integers as an equivalence relation). *Let $a, b, c, d, e, f \in \mathbb{N}$. Then*

(i) (reflexivity) $a —— b = b —— a$,

(ii) (symmetry) $a —— b = c —— d \implies c —— d = a —— b$, *and*

(iii) (transitivity) $a —— b = c —— d$ *and* $c —— d = e —— f \implies a —— b = e —— f$.

Definition 4.1.3 (Equivalence relations on sets). For objects R and X, we write "R is an equivalence relation on X" iff X is a set, $R \subseteq X \times X$, and for each $x, y, z \in X$, we have

(i) (reflexivity) $(x, x) \in R$,

(ii) (symmetry) $(x, y) \in R \implies (y, x) \in R$, and

(iii) (transitivity) $(x, y), (y, z) \in R \implies (x, z) \in R$.

Definition 4.1.4 (Equivalence relations)**.** An object R is called an equivalence relation iff there exists an object X such that R is an equivalence relation on X.

Remark 4.1.5. Axiom of substitution obeyed by both.

Lemma 4.1.6 (Domains of equivalence relations)**.** *Let R be an equivalence relation. Then there exists a unique set X such that R is an equivalence relation on X.*

Remark 4.1.7. This allows to denote X by dom_R. Axiom of substitution obeyed.

Lemma 4.1.8. *Let $P(R, X)$ be a property pertaining to any sets R, X. Then the following are equivalent:*

(i) Let R, X be sets such that R is an equivalence relation on X. Then $P(R, X)$.
(ii) Let R be an equivalence relation. Then $P(R, \mathrm{dom}_R)$.

Lemma 4.1.9. *Let R be an equivalence relation. Then* $\mathrm{dom}_R = \{p_1 : p \in R\}$.

Lemma 4.1.10 (Equivalence classes)**.** *Let R be an equivalence relation and $x \in \mathrm{dom}_R$. Then there exists a unique set Z such that for any object z, we have $z \in Z \iff (x, z) \in R$.*

Remark 4.1.11. This allows to denote Z by $[x]_R$. Axiom of substitution obeyed.

Lemma 4.1.12. *Let R be an equivalence relation and $x, y \in \mathrm{dom}_R$. Then $[x]_R = [y]_R \iff (x, y) \in R$.*

Definition 4.1.13 (Partitions of sets)**.** For objects C, X, we write "C is a partition on X" iff X is a set and

(i) $C \subseteq 2^X \setminus \{\emptyset\}$,
(ii) $\bigcup C = X$, and
(iii) for all sets $A, B \in C$, we have $A \neq B \implies A \cap B = \emptyset$.

Remark 4.1.14. Axiom of substitution obeyed.

Lemma 4.1.15 (Partitions induced by equivalence relations)**.** *Let R be an equivalence relation. Then there exists a unique set Z such that for any object A, we have $A \in Z \iff A = [x]_R$ for some $x \in \mathrm{dom}_R$. Further, such a set Z is a partition of* dom_R.

Remark 4.1.16. This allows to denote Z by part_R. Axiom of substitution obeyed.

June 4, 2021

Lemma 4.1.17 (Equivalence relation for integers)**.** *There exists a unique set Z such that for any object p, we have $p \in Z \iff p \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$ and $(p_1)_1 + (p_2)_2 = (p_2)_1 + (p_1)_2$. Further, such a set Z is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.*

Remark 4.1.18. This allows to denote Z be $R_{\mathbb{Z}}$.

Alternate definition 4.1.19 (Making integers, sets)**.**

(i) For any $a, b \in \mathbb{N}$, we set $a \text{—} b := [(a, b)]_{R_{\mathbb{Z}}}$.

(ii) An object p is called an integer iff $p = a \text{—} b$ for some $a, b \in \mathbb{N}$.

Remark 4.1.20. Axiom of substitution satisfied by both.

Remark 4.1.21. This does the following:

(i) Integers become sets, and their equality is now set equality.

(ii) Axioms 4.1 and 4.2 now become theorems

Lemma 4.1.22 (Set of integers)**.** *There exists a unique set X such that for any object x, we have $x \in X \iff x$ is an integer.*

Remark 4.1.23. This allows to denote X by $\mathbb{Z}$.

June 16, 2021

Lemma 4.1.24 (Addition on $\mathbb{Z}$)**.** *Let $p, q \in \mathbb{Z}$. Then there exists a unique $r \in \mathbb{Z}$ such that there exist $a, b, c, d \in \mathbb{N}$ such that $p = a \text{—} b$, and $q = c \text{—} d$ and $r = (a + c) \text{—} (b + d)$.*

Remark 4.1.25. Because of Axiom 4.2, the above lemma is equivalent to this: Let $p, q \in \mathbb{Z}$. Then there exists a unique $r \in \mathbb{Z}$ such that for each $a, b, c, d \in \mathbb{N}$, we have that $p = a \text{—} b$ and $q = c \text{—} d \implies r = a + c \text{—} b + d$.

Similar remarks for others.

Remark 4.1.26. This allows to denote r by $p + q$. Axiom of substitution obeyed.

Lemma 4.1.27. *Let $a, b, c, d, a', b', c', d' \in \mathbb{N}$ such that $a — b = a' — b'$ and $c — d = c' — d'$. Then $(ac + bd) — (ad + bc) = (a'c + b'd) — (a'd + b'c) = (a'c' + b'd') — (a'd' + b'c')$.*

Lemma 4.1.28 (Multiplication on $\mathbb{Z}$)**.** *Let $p, q \in \mathbb{Z}$. Then there exists a unique $r \in \mathbb{Z}$ such that there exist $a, b, c, d \in \mathbb{N}$ such that $p = a — b$, and $q = c — d$ and $r = (ac + bd) — (ad + bc)$.*

Remark 4.1.29. This allows to denote r by pq. Axiom of substitution obeyed.

Lemma 4.1.30 (Negation on $\mathbb{Z}$)**.** *Let $p \in \mathbb{Z}$. Then there exists a unique $q \in \mathbb{Z}$ such that there exist $a, b \in \mathbb{N}$ so that $p = a — b$ and $q = b — a$.*

Remark 4.1.31. This allows to denote q by $-p$. Axiom of substitution obeyed.

Corollary 4.1.32. *Let $a, b, c, d \in \mathbb{N}$. Then*

(i) $(a — b) + (c — d) = (a + c) — (b + d)$,
(ii) $(a — b)(c — d) = (ac + bd) — (ad + bc)$, *and*
(iii) $-(a — b) = b — a$.

Proposition 4.1.33 ($\mathbb{Z}$ forms a commutative ring)**.** *Let $p, q, r \in \mathbb{Z}$. Then*

$$\begin{aligned} p + q &= q + p, \\ (p + q) + r &= p + (q + r), \\ p + (0 — 0) &= p, \\ p + (-p) &= 0 — 0, \\ pq &= qp, \\ (pq)r &= p(qr), \\ p(1 — 0) &= p, \text{ and} \\ p(q + r) &= pq + pr. \end{aligned}$$

Remark 4.1.34. For any $p, q \in \mathbb{Z}$, we set $p - q := p + (-q)$. Negation precedes over subtraction.

Corollary 4.1.35. *Let $a, b \in \mathbb{N}$. Then $a — b = (a — 0) - (b — 0)$.*

Lemma 4.1.36. *Let $p, q \in \mathbb{Z}$. Then*

(i) $-p = (-(1 \text{——} 0))p$,
(ii) $-(p+q) = -p - q$,
(iii) $-(pq) = (-p)q$, *and*
(iv) $-(-p) = p$.

Proposition 4.1.37 ($\mathbb{Z}$ has no zero divisors)**.** *Let* $p, q \in \mathbb{Z}$ *such that* $pq = 0 \text{——} 0$. *Then* $p = 0 \text{——} 0$ *or* $q = 0 \text{——} 0$.

Corollary 4.1.38 (Cancellation law for multiplication on $\mathbb{Z}$)**.** *Let* $p, q, r \in \mathbb{Z}$ *such that* $r \neq 0 \text{——} 0$ *and* $pr = qr$. *Then* $p = q$.

Definition 4.1.39 (Positive and negative integers)**.**

(i) An object p is called a positive integer iff $p = n \text{——} 0$ for some positive natural n.
(ii) An object p is called a negative integer iff $p = 0 \text{——} n$ for some positive natural n.

Remark 4.1.40. Axiom of substitution obeyed.

Corollary 4.1.41 (Characterizing positive and negative integers)**.** *Let* $p \in \mathbb{Z}$. *Then*

(i) p *is a positive integer* $\iff$ $-p$ *is a negative integer, and*
(ii) p *is a negative integer* $\iff$ $-p$ *is a positive integer.*

Proposition 4.1.42 (Sums and products of positives are positive)**.** *Let* p, q *be positive integers. Then* $p + q$ *and* pq *are positive integers.*

Proposition 4.1.43 (Trichotomy for $\mathbb{Z}$)**.** *Let* $p \in \mathbb{Z}$. *Then exactly one of the following holds:*

(i) p *is a positive integer.*
(ii) $p = 0 \text{——} 0$.
(iii) p *is a negative integer.*

Definition 4.1.44 (Order on $\mathbb{Z}$)**.** For objects p, q, we write

(i) "$p > q$", or "$q < p$", iff $p, q \in \mathbb{Z}$ and $p - q$, is a positive integer, and
(ii) "$p \geq q$", or "$q \leq p$", iff $p, q \in \mathbb{Z}$ and $p - q = n \text{——} 0$ for some $n \in \mathbb{N}$.

Remark 4.1.45. Axiom of substitution obeyed. Strictly, a different symbol should've been used.

Corollary 4.1.46 (Characterizing "$\geq$" and "$>$")**.** *Let* $p, q \in \mathbb{Z}$. *Then*

(i) $p \geq q \iff p > q$ *or* $p = q$.
(ii) $p > q \iff p \geq q$ *and* $p \neq q$.

Lemma 4.1.47. *Let* $p, q \in \mathbb{Z}$. *Then* $p < q \iff p + (1—0) \leq q$.

Corollary 4.1.48 (Order properties for $\mathbb{Z}$)**.** *Let* $p, q, r \in \mathbb{Z}$. *Then*

(i) (transitivity) $p > q$ *and* $q > r \implies p > r$,
(ii) (addition preserves order) $p > q \implies p + r > q + r$,
(iii) (multiplication by positives preserves order) $p > q$ *and* r *is a positive integer* $\implies pr > qr$,
(iv) (negation reverses order) $p > q \implies -p < -q$, *and*
(v) (trichotomy) *exactly one of these holds:* $p < q$, $p = q$, *or* $p > q$.

Proposition 4.1.49 (Embedding $\mathbb{N}$ in $\mathbb{Z}$)**.** *There exists a unique function* f *such that* $f\colon \mathbb{N} \to \mathbb{Z}$ *so that* $f(n) = n—0$ *for each* $n \in \mathbb{N}$. *Further, for such a function* f, *for all* $m, n \in \mathbb{N}$, *we have*

(i) $m = n \iff f(m) = f(n)$,
(ii) $f(0) = 0—0$, *and*
(iii) $f(m{+}{+}) = f(m) + f(0{+}{+})$.

Remark 4.1.50. This allows to set $n_{\mathbb{Z}} \coloneqq f(n)$ for each $n \in \mathbb{N}$.

Corollary 4.1.51 (Properties of embedding)**.** *Let* $m, n \in \mathbb{N}$ *and* $p \in \mathbb{Z}$. *Then*

(i) $(m + n)_{\mathbb{Z}} = m_{\mathbb{Z}} + n_{\mathbb{Z}}$,
(ii) $(mn)_{\mathbb{Z}} = m_{\mathbb{Z}} n_{\mathbb{Z}}$,
(iii) m *is a positive natural* $\iff m_{\mathbb{Z}}$ *is a positive integer*,
(iv) $m > n \iff m_{\mathbb{Z}} > n_{\mathbb{Z}}$,
(v) $m \geq n \iff m_{\mathbb{Z}} \geq n_{\mathbb{Z}}$, *and*
(vi) *there exists a unique* $a \in \mathbb{N}$ *such that* $p = a_{\mathbb{Z}}$ *or* $p = -(a_{\mathbb{Z}})$.

Remark 4.1.52. This allows to throw off, in a sense, the symbol " — " since $m—n = m_{\mathbb{Z}} - n_{\mathbb{Z}}$ (Corollary 4.1.35).

Definition 4.1.53 (Even and odd integers)**.** An object p is called

(i) an even integer iff there exists a $q \in \mathbb{Z}$ such that $p = 2_{\mathbb{Z}} q$, and
(ii) an odd integer iff there exists a $q \in \mathbb{Z}$ such that $p = 2_{\mathbb{Z}} q + 1_{\mathbb{Z}}$.

Remark 4.1.54. Axiom of substitution obeyed.

Proposition 4.1.55 (Embedding consistent with odds and evens). *Let $m, n \in \mathbb{N}$. Then*

(i) m is an odd natural $\iff$ $f(m)$ is an odd integer, and
(ii) m is an even natural $\iff$ $f(m)$ is an even integer.

Proposition 4.1.56 (Properties of odds and evens). *Let $p \in \mathbb{Z}$. Then*

(i) p is an odd integer $\implies$ $p + 1_{\mathbb{Z}}$ and $p - 1_{\mathbb{Z}}$ are even integers,
(ii) p is an even integer $\implies$ $p + 1_{\mathbb{Z}}$ and $p - 1_{\mathbb{Z}}$ are odd integers, and
(iii) p is an even integer or an odd integer, but not both.

Proposition 4.1.57 (Well-ordering principle for $\mathbb{Z}$). *Let S be a set such that $S \subseteq \mathbb{Z}$, and $S \neq \emptyset$ and there exists an $m \in \mathbb{Z}$ such that $m \leq x$ for each $x \in S$. Then there exists a unique $p \in S$ such that $p \leq x$ for each $x \in S$.*

Remark 4.1.58. This allows to denote p by $\min(S)$. Axiom of substitution obeyed.

Lemma 4.1.59 (Embedding consistent with min). *Let S be a set such that $S \subseteq \mathbb{N}$ and $S \neq \emptyset$ and set $S' := \{n_{\mathbb{Z}} : n \in S\}$. Then $S' \subseteq \mathbb{Z}$, and $S' \neq \emptyset$ and there exists an $m \in \mathbb{Z}$ such that $m \leq x$ for each $x \in S'$, and we also have* $\min(S)_{\mathbb{Z}} = \min(S')$.

4.2 The rationals

June 19, 2021

Axiom 4.3 (Rationals). *Rationals are objects.*

Axiom 4.4 (Properties of the function symbol "$a//b$"). *(i) Let $a, b \in \mathbb{Z}$ such that $b \neq 0_{\mathbb{Z}}$. Then $a//b$ is a rational.*
(ii) Let $a, b, c, d \in \mathbb{Z}$ such that $b, d \neq 0_{\mathbb{Z}}$. Then $a//b = c//d \iff ad = cb$.
(iii) Let r be a rational. Then $r = a//b$ for some $a, b \in \mathbb{Z}$ such that $b \neq 0_{\mathbb{Z}}$.

Remark 4.2.1. The domain of the function symbol " // " is integers in the first slot and non-zero integers in the second slot.

Lemma 4.2.2 (Consistency of equality of rationals as an equivalence relation). *Let $a, b, c, d, e, f \in \mathbb{Z}$ such that $b, d, f \neq 0_{\mathbb{Z}}$. Then*

(i) (reflexivity) *$a//b = a//b$,*

(ii) (symmetry) $a//b = c//d \implies c//d = a//b$, *and*
(iii) (transitivity) $a//b = c//d$ *and* $c//d = e//f \implies a//b = e//f$.

Lemma 4.2.3 (Equivalence relation for rationals)**.** *There exists a unique set X such that for any object p, we have $p \in X \iff p \in (\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\}) \times (\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\}))$ and $(p_1)_1(p_2)_2 = (p_2)_1(p_1)_2$. Further, such a set X is an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\})$.*

Remark 4.2.4. This allows to denote X by $R_{\mathbb{Q}}$.

Alternate definition 4.2.5 (Making rationals, sets)**.**
(i) For any $a \in \mathbb{Z}$, any $b \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$, we set $a//b := [(a, b)]_{R_{\mathbb{Q}}}$.
(ii) An object r is called a rational iff $r = a//b$ for some $a \in \mathbb{Z}$ and some $b \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$.

Remark 4.2.6. Axiom of substitution obeyed.

Remark 4.2.7. This does the following:
(i) Rationals become sets, and their equality becomes set equality.
(ii) Axioms 4.3 and 4.4 now become theorems.

Lemma 4.2.8 (Set of rationals)**.** *There exists a unique set X such that for any object r, we have $r \in X \iff r$ is a rational.*

Remark 4.2.9. This allows to denote X by $\mathbb{Q}$.

Lemma 4.2.10 (Addition on $\mathbb{Q}$)**.** *Let $r, s \in \mathbb{Q}$. Then there exists a unique $t \in \mathbb{Q}$ such that there exist $a, b, c, d \in \mathbb{Z}$, so that $b, d, bd \neq 0_{\mathbb{Z}}$, and $r = a//b$, and $s = c//d$ and $t = (ad + cb)//(bd)$.*

Remark 4.2.11. This allows to denote t by $r + s$. Axiom of substitution obeyed.

Lemma 4.2.12 (Multiplication on $\mathbb{Q}$)**.** *Let $r, s \in \mathbb{Q}$. Then there exists a unique $t \in \mathbb{Q}$ such that there exist $a, b, c, d \in \mathbb{Z}$, so that $b, d, bd \neq 0_{\mathbb{Z}}$, and $r = a//b$, and $s = c//d$ and $t = (ac)//(bd)$.*

Remark 4.2.13. This allows to denote t by rs. Axiom of substitution obeyed.

Lemma 4.2.14 (Negation on $\mathbb{Q}$)**.** *Let $r \in \mathbb{Q}$. Then there exists a unique $s \in \mathbb{Q}$ such that there exist $a, b \in \mathbb{Z}$ so that $b \neq 0_{\mathbb{Z}}$, and $r = a//b$ and $s = (-a)//b$.*

Remark 4.2.15. This allows to denote s by $-r$. Axiom of substitution obeyed.

Lemma 4.2.16. *Let $a, b \in \mathbb{Z}$ and $b \neq 0_\mathbb{Z}$. Then $a//b = 0_\mathbb{Z}//1_\mathbb{Z} \iff a = 0_\mathbb{Z}$.*

Lemma 4.2.17 (Reciprocation on $\mathbb{Q}$)**.** *Let $r \in \mathbb{Q}$ such that $r \neq 0_\mathbb{Z}//1_\mathbb{Z}$. Then there exists a unique $s \in \mathbb{Q}$ such that there exist $a, b \in \mathbb{Z}$ so that $a, b \neq 0_\mathbb{Z}$, and $r = a//b$ and $s = b//a$.*

Remark 4.2.18. This allows to denote s by r^{-1}. Axiom of substitution obeyed.

Corollary 4.2.19. *Let $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0_\mathbb{Z}$. Then $bd \neq 0_\mathbb{Z}$, and*

(i) $(a//b) + (c//d) = (ad + cb)//(bd)$,
(ii) $(a//b)(c//d) = (ac)//(bd)$,
(iii) $-(a//b) = (-a)//b$, *and*
(iv) $a \neq 0_\mathbb{Z} \implies a//b \neq 0_\mathbb{Z}//1_\mathbb{Z}$ *and* $(a//b)^{-1} = b//a$.

Proposition 4.2.20 ($\mathbb{Q}$ forms a field)**.** *Let $r, s, t \in \mathbb{Q}$. Then*

$$\begin{aligned} r + s &= s + r, \\ (r + s) + t &= r + (s + t), \\ r + (0_\mathbb{Z}//1_\mathbb{Z}) &= r, \\ r + (-r) &= 0_\mathbb{Z}//1_\mathbb{Z}, \\ rs &= sr, \\ (rs)t &= r(st), \\ r(1_\mathbb{Z}//1_\mathbb{Z}) &= r, \\ rr^{-1} &= 1_\mathbb{Z}//1_\mathbb{Z} \qquad \text{if } r \neq 0_\mathbb{Z}//1_\mathbb{Z}, \text{ and} \\ r(s + t) &= rs + rt. \end{aligned}$$

Remark 4.2.21. For any $r, s \in \mathbb{Q}$, we set $r - s := r + (-s)$. Further, if $s \neq 0_\mathbb{Z}//1_\mathbb{Z}$, we set $r/s := rs^{-1}$.

Corollary 4.2.22. *Let $p, q \in \mathbb{Z}$ such that $q \neq 0_\mathbb{Z}$. Then $q//1_\mathbb{Z} \neq 0_\mathbb{Z}//1_\mathbb{Z}$ and $p//q = (p//1_\mathbb{Z})/(q//1_\mathbb{Z})$.*

Lemma 4.2.23. *Let $r, s \in \mathbb{Q}$. Then the analogue of Lemma 4.1.36 holds. Further, if $r, s \neq 0_\mathbb{Z}//1_\mathbb{Z}$, then $rs, -r, r^{-1} \neq 0_\mathbb{Z}//1_\mathbb{Z}$, and*

(i) $(rs)^{-1} = r^{-1}s^{-1}$,

(ii) $(-r)^{-1} = -(r^{-1})$, *and*
(iii) $(r^{-1})^{-1} = r$.

Definition 4.2.24 (Positive and negative rationals)**.**
(i) An object r is called a positive rational iff $r = p//q$ for some positive integers p, q.
(ii) An object r is called a negative rational iff $r = a//b$ for some negative integer a and some positive integer b.

Remark 4.2.25. Axiom of substitution obeyed.

Corollary 4.2.26 (Characterization of positive and negative rationals)**.** *The analogue of Corollary 4.1.41 holds.*

Proposition 4.2.27 (Sums and products of positives are positive)**.** *Let r, s be positive rationals. Then $r + s$ and rs are positive rationals.*

Proposition 4.2.28 (Trichotomy for $\mathbb{Q}$)**.** *Let $r \in \mathbb{Q}$. Then exactly one of the following holds:*
(i) r *is a positive rational.*
(ii) $r = 0_{\mathbb{Z}}//1_{\mathbb{Z}}$.
(iii) r *is a negative rational.*

Definition 4.2.29 (Order on $\mathbb{Q}$)**.** For objects r, s, we write
(i) "$r > s$", or "$s < r$", iff $r, s \in \mathbb{Q}$ and $r - s$ is a positive rational, and
(ii) "$r \geq s$", or "$s \leq r$", iff $r, s \in \mathbb{Q}$ and $r - s = p//q$ for some positive integer q and some integer $p \geq 0_{\mathbb{Z}}$.

Remark 4.2.30. Axiom of substitution obeyed. Again a different symbol should have been used.

Corollary 4.2.31 (Characterizing " $\geq$ " and " $>$ ")**.** *The analogue of Corollary 4.1.46 holds.*

Corollary 4.2.32 (Order properties for $\mathbb{Q}$)**.** *Let $r, s, t \in \mathbb{Q}$. Then along with the analogue of Corollary 4.1.48, the following hold:*
(i) (reciprocation preserves positiveness) r *is a positive rational* $\implies$ $r \neq 0_{\mathbb{Z}}//1_{\mathbb{Z}}$ *and* r^{-1} *is a positive rational.*
(ii) (positive reciprocation reverses order) $r > s > 0_{\mathbb{Z}}//1_{\mathbb{Z}} \implies r, s \neq 0_{\mathbb{Z}}//1_{\mathbb{Z}}$ *and* $s^{-1} > r^{-1} > 0_{\mathbb{Z}}//1_{\mathbb{Z}}$.

Proposition 4.2.33 (ε-characterization of " $\leq$ "). *Let $x, y \in \mathbb{Q}$. Then $x \leq y \iff x \leq y + \varepsilon$ for each $\varepsilon > 0_{\mathbb{Q}}$.*

Proposition 4.2.34 (Embedding $\mathbb{Z}$ into $\mathbb{Q}$). *There exists a unique function f such that $f : \mathbb{Z} \to \mathbb{Q}$ so that $f(p) = p//1_{\mathbb{Z}}$ for each $p \in \mathbb{Z}$. Further, for such an f, for any $p, q \in \mathbb{Z}$, we have*

(i) $p = q \iff f(p) = f(q)$,
(ii) $f(p + q) = f(p) + f(q)$, *and*
(iii) $f(pq) = f(p)f(q)$.

Remark 4.2.35. This allows to set $p_{\mathbb{Q}'} := f(p)$ for each $p \in \mathbb{Z}$.

Proposition 4.2.36 (Properties of embedding). *Let $p, q \in \mathbb{Z}$ and $r \in \mathbb{Q}$. Then*

(i) $(-p)_{\mathbb{Q}'} = -p_{\mathbb{Q}'}$,
(ii) $(p - q)_{\mathbb{Q}'} = p_{\mathbb{Q}'} - q_{\mathbb{Q}'}$,
(iii) *p is a positive integer $\iff$ $p_{\mathbb{Q}'}$ is a positive rational,*
(iv) *p is a negative integer $\iff$ $p_{\mathbb{Q}'}$ is a negative rational,*
(v) $p > q \iff p_{\mathbb{Q}'} > q_{\mathbb{Q}'}$,
(vi) $p \geq q \iff p_{\mathbb{Q}'} \geq q_{\mathbb{Q}'}$, *and*
(vii) *there exist $a, b \in \mathbb{Z}$ such that $b \neq 0_{\mathbb{Q}'}$ and $r = a_{\mathbb{Q}'}/b_{\mathbb{Q}'}$.*

Remark 4.2.37. This allows to throw off, in a sense, the symbol " // " since $p//q = p_{\mathbb{Q}'}/q_{\mathbb{Q}'}$ for non-zero q (Corollary 4.2.22).

Remark 4.2.38. For each $n \in \mathbb{N}$, we set $n_{\mathbb{Q}} := (n_{\mathbb{Z}})_{\mathbb{Q}'}$. The common properties hold.

4.3 Absolute value and exponentiation

June 23, 2021

Remark 4.3.1. This section will hold for all the ordered fields and not just $\mathbb{Q}$. (We can't conclude anything for $\mathbb{C}$ for instance.)

Lemma 4.3.2 (Some further order properties for $\mathbb{Q}$). *Let $x, y, z, w \in \mathbb{Q}$. Then*

(i) $x < y$ *and* $z < w \implies x + z < y + w$, *and*
(ii) $0_{\mathbb{Q}} < x < y$ *and* $0_{\mathbb{Q}} < z < w \implies 0_{\mathbb{Q}} < xz < yw$.

Lemma 4.3.3 (Absolute value for $\mathbb{Q}$). *Let $x \in \mathbb{Q}$. Then there exists a unique $y \in \mathbb{Q}$ such that one of the following holds:*

(i) x *is positive and* $y = x$.
(ii) $x = 0_{\mathbb{Q}}$ *and* $y = 0_{\mathbb{Q}}$.
(iii) x *is negative and* $y = -x$.

Further, for any $y \in \mathbb{Q}$, at most one of the above holds.

Remark 4.3.4. This allows to denote y by $|x|$. Axiom of substitution obeyed.

Proposition 4.3.5 (Properties of absolute values). *Let $x, y, z \in \mathbb{Q}$. Then*

(i) (non-negativity) $|x| \geq 0_{\mathbb{Q}}$,
(ii) (non-degeneracy) $|x| = 0_{\mathbb{Q}} \iff x = 0_{\mathbb{Q}}$, *and*
(iii) (absolute values of products) $|xy| = |x||y|$, *and*
(iv) (triangle inequality) $|x + y| \leq |x| + |y|$.

Corollary 4.3.6 (Further properties of absolute values). *Let $x, y, \varepsilon \in \mathbb{Q}$. Then*

(i) (absolute values of negations) $|-x| = |x|$,
(ii) (absolute values of reciprocals) $x \neq 0_{\mathbb{Q}} \implies |x| \neq 0_{\mathbb{Q}}$ *and* $|x^{-1}| = |x|^{-1}$
(iii) $||x| - |y|| \leq |x + y|$,
(iv) $|x - y| \leq \delta$ *for each* $\delta > 0_{\mathbb{Q}} \iff x = y$,
(v) $y > |x| \iff -y < x < y$,
(vi) $y \geq |x| \iff -y \leq x \leq y$ *and hence,* $-|x| \leq x \leq |x|$,
(vii) $|x - y| < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$, *and*
(viii) $|x - y| \leq \varepsilon \iff y - \varepsilon \leq x \leq y + \varepsilon$.

Remark 4.3.7. For any $x, y \in \mathbb{Q}$, we set $d(x, y) := |x - y|$.

Corollary 4.3.8 (Properties of distances). *Let $, y, z \in \mathbb{Q}$. Then*

(i) (non-negativity) $d(x, y) \geq 0_{\mathbb{Q}}$,
(ii) (non-degeneracy) $d(x, y) = 0_{\mathbb{Q}} \iff x = y$, *and*
(iii) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

Remark 4.3.9. We'll abbreviate "let $\varepsilon \in \mathbb{Q}$ such that $\varepsilon > 0_{\mathbb{Q}}$" by "let $\varepsilon > 0_{\mathbb{Q}}$" for the results, not for axioms or definitions. (Extensible to other possible generalizations.)

Proposition 4.3.10 (Further properties of distances on $\mathbb{Q}$). *Let $x, y, z, w \in \mathbb{Q}$ and $\varepsilon, \delta > 0_{\mathbb{Q}}$. Then the following hold:*

(i) $d(x, y) \leq \tilde{\varepsilon}$ *for each* $\tilde{\varepsilon} > 0_{\mathbb{Q}} \iff x = y$.
(ii) $d(x, y) \leq \varepsilon$ *and* $d(y, z) \leq \delta \implies d(x, z) \leq \varepsilon + \delta$.
(iii) $d(x, y) \leq \varepsilon$ *and* $d(z, w) \leq \delta \implies d(x + z, y + w) \leq \varepsilon + \delta$.
(iv) $d(x, y) \leq \varepsilon \implies d(xz, yz) \leq \varepsilon|z|$.
(v) $d(x, y) \leq \varepsilon$ *and* $d(z, w) \leq \delta \implies d(xz, yw) \leq \varepsilon|z| + \delta|x| + \varepsilon\delta$.
(vi) $d(y, x), d(z, x) \leq \varepsilon$ *and* $y \leq w \leq z \implies d(w, x) \leq \varepsilon$.

Lemma 4.3.11 (Exponentiation of rationals by naturals)**.** *Let* $x \in \mathbb{Q}$. *Then there exists a unique function* $f : \mathbb{N} \to \mathbb{Q}$ *such that* $f(0) = 1_{\mathbb{Q}}$ *and* $f(n+1) = f(n)x$ *for each* $n \in \mathbb{N}$.

Remark 4.3.12. This allows to denote $f(n)$ by x^n for each $n \in \mathbb{N}$. Axiom of substitution obeyed.

Proposition 4.3.13. *Let* $n \in \mathbb{N}$ *such that* $n \geq 1$. *Then*

(i) (a) n *is odd* $\implies (-1_{\mathbb{Q}})^n = -1_{\mathbb{Q}}$,
  (b) n *is even* $\implies (-1_{\mathbb{Q}})^n = 1_{\mathbb{Q}}$,
(ii) $(0_{\mathbb{Q}})^n = 0_{\mathbb{Q}}$, *and*
(iii) $(1_{\mathbb{Q}})^n = 1_{\mathbb{Q}}$.

Proposition 4.3.14 (Properties of exponentiation by naturals)**.** *Let* $x, y, z \in \mathbb{Q}$ *such that* $z \neq 0_{\mathbb{Q}}$. *Let* $m, n \in \mathbb{N}$ *and* $k \geq 1$. *Then* $z^n \neq 0_{\mathbb{Q}}$

(i) $(xy)^n = x^n y^n$,
(ii) $x^{m+n} = x^m x^n$,
(iii) $x^{mn} = (x^m)^n$,
(iv) (a) n *is odd* $\implies (-x)^n = -x^n$,
  (b) n *is even* $\implies (-x)^n = x^n$,
(v) $(z^{-1})^n = (z^n)^{-1}$,
(vi) (a) $x^n = 0_{\mathbb{Q}} \iff x = 0_{\mathbb{Q}}$ *and* $n > 0$,
  (b) x *is a positive rational* $\implies x^n$ *is a positive rational*,
(vii) $x > y > 0_{\mathbb{Q}} \implies x^k > y^k > 0_{\mathbb{Q}}$, *and*
(viii) $|x|^n = |x^n|$.

Corollary 4.3.15.

(i) *Let* $x, y \geq 0_{\mathbb{Q}}$ *and* n *be a positive natural such that* $x^n = y^n$. *Then* $x = y$.
(ii) *Let* $x \in \mathbb{Q}$ *and* $m, n \in \mathbb{N}$ *such that* $m < n$. *Then*
  (a) $0_{\mathbb{Q}} < x < 1_{\mathbb{Q}} \implies x^m > x^n$ *and,*
  (b) $x > 1_{\mathbb{Q}} \implies x^n > x^m$.

Lemma 4.3.16 (Exponentiation of non-zero rationals by integers)**.** *Let $x \in \mathbb{Q}$ and $p \in \mathbb{Z}$ such that $x \neq 0_{\mathbb{Q}}$. Then there exists a unique $y \in \mathbb{Q}$ such that there exists a positive natural n so that one of the following holds:*

(i) $p = n_{\mathbb{Z}}$ *and* $y = x^n$.
(ii) $p = 0_{\mathbb{Z}}$ *and* $y = 1_{\mathbb{Q}}$.
(iii) $p = -n_{\mathbb{Z}}$, *and* $x^n \neq 0_{\mathbb{Q}}$ *and* $y = (x^n)^{-1}$.

Further, for any $x, y \in \mathbb{Q}$, for any $p \in \mathbb{Z}$ and for any positive natural, at most one of the above holds.

Remark 4.3.17. This allows to denote y by x^p. Axiom of substitution obeyed.

Corollary 4.3.18. *Let $x \in \mathbb{Q}$ and $n \in \mathbb{N}$ such that $x \neq 0_{\mathbb{Q}}$. Then*

(i) $x^{n_{\mathbb{Z}}} = x^n$, *and*
(ii) $x^n \neq 0_{\mathbb{Q}}$ *and* $x^{-n_{\mathbb{Z}}} = (x^n)^{-1}$, *and in particular,* $x^{-1_{\mathbb{Z}}} = x^{-1}$.

Proposition 4.3.19. *Let $p \in \mathbb{Z}$. Then*

(i) *(a)* p *is odd* $\implies (-1_{\mathbb{Q}})^p = -1_{\mathbb{Q}}$,
(b) p *is even* $\implies (-1_{\mathbb{Q}})^p = 1_{\mathbb{Q}}$, *and*
(ii) $(1_{\mathbb{Q}})^p = 1_{\mathbb{Q}}$.

Lemma 4.3.20. *Let $x \in \mathbb{Q}$ and $p \in \mathbb{Z}$ such that $x \neq 0_{\mathbb{Q}}$. Then $x^p \neq 0_{\mathbb{Q}}$.*

Proposition 4.3.21 (Properties of exponentiation of non-zero rationals by integers)**.** *Let $x, y \in \mathbb{Q}$, and $p, q \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $x, y \neq 0_{\mathbb{Q}}$. Then $xy, x^p, -x, x^{-1}, |x|, x^n \neq 0_{\mathbb{Q}}$, and*

(i) $(xy)^p = x^p y^p$,
(ii) $x^{p+q} = x^p x^q$,
(iii) $x^{pq} = (x^p)^q$,
(iv) *(a)* p *is odd* $\implies (-x)^p = -x^p$,
(b) p *is even* $\implies (-x)^p = x^p$,
(v) $(x^{-1})^p = (x^p)^{-1} = x^{-p}$,
(vi) x *is a positive rational* $\implies x^p$ *is a positive rational,*
(vii) $p > 0_{\mathbb{Z}}$ *and* $x > y > 0_{\mathbb{Q}} \implies x^p > y^p > 0_{\mathbb{Q}}$,
(viii) $|x|^p = |x^p|$, *and*
(ix) $(x^n)^p = (x^p)^n = x^{n_{\mathbb{Z}} p}$.

Corollary 4.3.22. *(i) Let $x, y > 0_{\mathbb{Q}}$, and $p \in \mathbb{Z}$ such that $p \neq 0_{\mathbb{Z}}$, and $x^p = y^p$. Then $x = y$.*

(ii) *Let* $x \in \mathbb{Q}$ *and* $p, q \in \mathbb{Z}$ *such that* $p < q$. *Then*
  (a) $0_\mathbb{Q} < x < 1_\mathbb{Q} \implies x^p > x^q$ *and,*
  (b) $x > 1_\mathbb{Q} \implies x^q > x^p$.

Lemma 4.3.23 (Embedding consistent with exponentiation)**.** *Let* $m, n \in \mathbb{N}$. *Then* $(m_\mathbb{Q})^n = (m^n)_\mathbb{Q}$.

Lemma 4.3.24. *Let* $n \in \mathbb{N}$. *Then* $2^n > n$.

Corollary 4.3.25. *Let* $N \in \mathbb{Z}$ *such that* $N \geq 0_\mathbb{Z}$. *Then* $(2_\mathbb{Q})^N > N_{\mathbb{Q}'}$.

4.4 Gaps in the rational numbers

June 23, 2021

Lemma 4.4.1 (Rationals surrounded by rationals)**.** *Let* $x \in \mathbb{Q}$ *and* $\varepsilon > 0_\mathbb{Q}$. *Then there exist* $r, s \in \mathbb{Q}$ *such that* $r < x < s$ *and* $s - r < \varepsilon$.

Lemma 4.4.2 ($\mathbb{Q}$ is unbounded)**.** *Let* $x > 0_\mathbb{Q}$. *Then there exists a* $y > 0_\mathbb{Q}$ *such that* $x < y$.

Lemma 4.4.3 ($\mathbb{N}$ is cofinal in $\mathbb{Q}$)**.** *Let* $r \in \mathbb{Q}$. *Then there exists an* $N \in \mathbb{N}$ *such that* $r < N_\mathbb{Q}$.

Corollary 4.4.4. *(i)* (Archimedean property of $\mathbb{Q}$)**.** *Let* $\varepsilon > 0_\mathbb{Q}$ *and* $N \in \mathbb{N}$. *Then there exists an* $M \in \mathbb{N}$ *such that* $M_\mathbb{Q}\varepsilon > N_\mathbb{Q}$.

(ii) *Let* $r > 0_\mathbb{Q}$. *Then there exists an* $N \geq 1$ *such that* $N_\mathbb{Q} \neq 0_\mathbb{Q}$ *and* $r > 1_\mathbb{Q}/N_\mathbb{Q} > 0_\mathbb{Q}$.

(iii) (Floor function for $\mathbb{Q}$)**.** *Let* $r \in \mathbb{Q}$. *Then there exists a unique* $p \in \mathbb{Z}$ *such that* $p_{\mathbb{Q}'} \leq r < (p + 1_\mathbb{Z})_{\mathbb{Q}'}$.

(iv) (Rationals between rationals)**.** *Let* $x, y \in \mathbb{Q}$ *such that* $x < y$. *Then there exists an* $r \in \mathbb{Q}$ *such that* $x < r < y$.

Remark 4.4.5. (iii) allows to denote p by $\lfloor r \rfloor$. Axiom of substitution obeyed.

Lemma 4.4.6. *Let* $r \geq -1_\mathbb{Q}$ *and* $m \in \mathbb{N}$. *Then* $(1_\mathbb{Q} + r)^m \geq 1_\mathbb{Q} + m_\mathbb{Q} r$.

Corollary 4.4.7 (Archimedean property for natural exponentiation on $\mathbb{Q}$)**.** *Let* $r > 1_\mathbb{Q}$ *and* $N \in \mathbb{N}$. *Then there exists an* $m \in \mathbb{N}$ *such that* $r^m > N_\mathbb{Q}$.

Proposition 4.4.8 (Square of no rational equals 2)**.** *Let* $r \in \mathbb{Q}$. *Then* $r^2 \neq 2_\mathbb{Q}$.

Proposition 4.4.9 (Squares of rationals get arbitrarily close to 2)**.** *Let* $\varepsilon > 0_\mathbb{Q}$. *Then there exists a non-negative rational* r *such that* $r^2 < 2_\mathbb{Q} < (r + \varepsilon)^2$.

Chapter 5

The real numbers

5.1 Cauchy sequences

June 30, 2021

Definition 5.1.1 (Sequences of rationals). (i) For any objects a, m, n, we write "$(a_i)_{i=m}^{n}$ is a sequence of rationals" iff $m, n \in \mathbb{Z}$ and $a\colon \{i \in \mathbb{Z} : m \le i \le n\} \to \mathbb{Q}$.

(ii) For any objects a, m, we write "$(a_i)_{i=m}^{\infty}$ is a sequence of rationals" iff $m \in \mathbb{Z}$ and $a\colon \{i \in \mathbb{Z} : i \ge m\} \to \mathbb{Q}$.

Remark 5.1.2. Axiom of substitution obeyed by both.

Definition 5.1.3 (Cauchy sequences of rationals). For any objects a, m, we write "$(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of rationals" iff $(a_i)_{i=m}^{\infty}$ is a sequence of rationals and for each $\varepsilon > 0_{\mathbb{Q}}$, there exists an $N \ge m$ such that for all $i, j \ge N$, we have $|a_i - a_j| \le \varepsilon$.

Remark 5.1.4. Axiom of substitution obeyed.

Proposition 5.1.5 ($(1/n)_{n=1}^{\infty}$ is a Cauchy sequence). *There exists a unique function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a sequence of rationals and for each $i \ge 1_{\mathbb{Z}}$, we have $i_{\mathbb{Q}'} \ne 0_{\mathbb{Q}}$ and $a_i = (i_{\mathbb{Q}'})^{-1}$. Further, for any such a, we have that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals.*

Definition 5.1.6 (Bounded sequences). (i) For any objects a, m, n, we write "$(a_i)_{i=m}^{n}$ is a bounded sequence of rationals" iff $(a_i)_{i=m}^{n}$ is a sequence of rationals and there exists an object M such that for each $m \le i \le n$, we have $|a_i| \le M$.

(ii) For any objects a, m, we write "$(a_i)_{i=m}^{\infty}$ is a bounded sequence of rationals" iff $(a_i)_{i=m}^{\infty}$ is a sequence of rationals and there exists an object M such that for each $i \geq m$, we have that $|a_i| \leq M$.

Remark 5.1.7. Axiom of substitution obeyed by both.

Lemma 5.1.8. *Let a be a function, $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $(a_i)_{i=m}^{m+n_{\mathbb{Z}}}$ is a sequence of rationals. Then $(a_i)_{i=m}^{m+n_{\mathbb{Z}}}$ is a bounded sequence of rationals.*

Corollary 5.1.9 (Finite sequences of rationals are bounded)**.** *Let a be a function and $m, n \in \mathbb{Z}$ such that $(a_i)_{i=m}^{n}$ is a sequence of rationals. Then $(a_i)_{i=m}^{n}$ is a bounded sequence of rationals.*

Lemma 5.1.10 (Cauchy sequences of rationals are bounded)**.** *Let a be a function and $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of rationals. Then $(a_i)_{i=m}^{\infty}$ is a bounded sequence of rationals.*

5.2 Equivalent Cauchy sequences

Definition 5.2.1 (Equivalent sequences of rationals)**.** For objects a, b, m, n, we write "$(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=n}^{\infty}$ are equivalent sequences of rationals" iff $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=n}^{\infty}$ are sequences of rationals, and $m = n$, and for each $\varepsilon > 0_{\mathbb{Q}}$, there exists an $N \geq m, n$ such that for each $i \geq N$, we have that $|a_i - b_i| \leq \varepsilon$.

Remark 5.2.2. Axiom of substitution obeyed.

Lemma 5.2.3 (Rational sequences that finitely differ are equivalent)**.** *Let a, b be functions and $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ are sequences of rationals and there exists an $N \geq m$ such that for each $i \geq N$, we have $a_i = b_i$. Then $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=m}^{\infty}$ are equivalent sequences of rationals.*

Proposition 5.2.4. *Let $x, y \in \mathbb{Q}$ such that $|y| > 1_{\mathbb{Q}}$. Then there exist unique functions a, b such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are sequences of rationals and $a_n = x + y^{-n}$ and $b_n = x - y^{-n}$ for each $n \geq 1_{\mathbb{Z}}$. Further, for any such functions a, b, we have that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ and $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are equivalent sequences of rationals.*

Proposition 5.2.5 (Properties of equivalent sequences of rationals)**.** *Let a, b be functions and $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=m}^{\infty}$ are equivalent sequences of rationals. Then* *(i)* $(a_i)_{i=m}^{\infty}$ *is a Cauchy sequence of rationals* $\iff$ $(b_i)_{i=m}^{\infty}$ *is a Cauchy sequence of rationals, and*

(ii) $(a_i)_{i=m}^{\infty}$ *is a bounded sequence of rationals* $\iff$ $(b_i)_{i=m}^{\infty}$ *is a bounded sequence of rationals.*

5.3 The construction of the real numbers

Axiom 5.1 (Reals)**.** *Reals are objects.*

Axiom 5.2 (Properties of the function symbol "$\mathrm{LIM}_{n\to\infty} a_n$")**.** *(i) Let a be an object such that* $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *is a Cauchy sequence of rationals. Then* $\mathrm{LIM}_{n\to\infty} a_n$ *is a real.*

(ii) Let a, b be objects such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *and* $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *are Cauchy sequences of rationals. Then* $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n \iff (a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *and* $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *are equivalent sequences of rationals.*

(iii) Let x be a real. Then $x = \mathrm{LIM}_{n\to\infty} a_n$ *for some object a such that* $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *is a Cauchy sequence of rationals.*

Remark 5.3.1. Domain of the function symbol "$\mathrm{LIM}_{n\to\infty}\ {}_n$" is any function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals.

Lemma 5.3.2 (Consistency of the equality of reals as an equivalence relation)**.** *Let a, b, c be functions such that* $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$*,* $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$*,* $(c_i)_{i=1}^{\infty}$ *are Cauchy sequences of rationals. Then*

(i) (reflexivity) $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} a_n$*,*

(ii) (symmetry) $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n \implies \mathrm{LIM}_{n\to\infty} b_n = \mathrm{LIM}_{n\to\infty} a_n$*, and*

(iii) (transitivity) $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$ *and* $\mathrm{LIM}_{n\to\infty} b_n = \mathrm{LIM}_{n\to\infty} c_n \implies \mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} c_n$

Remark 5.3.3. We set $A \coloneqq \{a \in \mathbb{Q}^{\{i\in\mathbb{Z}: i\geq 1_{\mathbb{Z}}\}} : (a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals$\}$ to be used in Lemma 5.3.4 and Alternate Definition 5.3.6.

Lemma 5.3.4 (Equivalence relation for reals)**.** *There exists a unique set X such that for any object p, we have* $p \in X \iff p \in A \times A$ *and* $((p_1)_i)_{i=1_{\mathbb{Z}}}^{\infty}$*,* $((p_2)_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *are equivalent sequences of rationals. Further, such a set X is an equivalence relation on A.*

Remark 5.3.5. This allows to denote X by $R_{\mathbb{R}}$.

Alternate definition 5.3.6 (Making reals, sets)**.** (i) For any $a \in A$, we set $\mathrm{LIM}_{n\to\infty} a_n := [a]_{R_\mathbb{R}}$.
(ii) An object x is called a real iff $x = \mathrm{LIM}_{n\to\infty} a_n$ for some $a \in A$.

Remark 5.3.7. Axiom of substitution obeyed.

Remark 5.3.8. This does the following:

(i) Reals become sets, and their equality becomes set equality.
(ii) Axioms 5.1 and 5.2 now become theorems.

Lemma 5.3.9 (Set of reals)**.** *There exists a unique set X such that for any object x, we have $x \in X \iff x$ is a real.*

Lemma 5.3.10 (Beginning integers of rational sequences)**.** *Let a be a function such that there exists an $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^\infty$ is a sequence of rationals. Then there exists a unique $n \in \mathbb{Z}$ such that $(a_i)_{i=n}^\infty$ is a sequence of rationals.*

Remark 5.3.11. This allows to denote n be $\mathrm{SeqInt}_\mathbb{Q}\, a$. Axiom of substitution obeyed.

Lemma 5.3.12 (Sums and products of sequences of rationals)**.** *Let a, b be functions such that there exists an $m \in \mathbb{Z}$ so that $(a_i)_{i=m}^\infty$ and $(b_i)_{i=m}^\infty$ are sequences of rationals. Then $\mathrm{SeqInt}_\mathbb{Q}\, a = \mathrm{SeqInt}_\mathbb{Q}\, b$ and there exist unique functions c, d such that $(c_i)_{i=\mathrm{SeqInt}_\mathbb{Q} a}^\infty$, $(d_i)_{i=\mathrm{SeqInt}_\mathbb{Q} a}^\infty$ are sequences of rationals and for each $i \geq \mathrm{SeqInt}_\mathbb{Q}\, a$, we have $c_i = a_i + b_i$ and $d_i = a_i b_i$.*

Remark 5.3.13. This allows to denote c and d by $a + b$ and ab respectively. Axiom of substitution obeyed.

Lemma 5.3.14 (Negation of sequences of rationals)**.** *Let a be a function such that there exists an $m \in \mathbb{Z}$ so that $(a_i)_{i=m}^\infty$ is a sequence of rationals. Then there exists a unique function c such that $(c_i)_{i=\mathrm{SeqInt}_\mathbb{Q} a}^\infty$ is a sequence of rationals and for each $i \geq \mathrm{SeqInt}_\mathbb{Q}\, a$, we have $c_i = -(a_i)$.*

Remark 5.3.15. This allows to denote c by $-a$. Axiom of substitution obeyed.

Lemma 5.3.16 (Reciprocation of non-zero sequences of rationals)**.** *Let a be a function such that there exists an $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^\infty$ is a sequence of rationals and for each $i \geq m$, we have $a_i \neq 0_\mathbb{Q}$. Then there exists a unique function c such that $(c_i)_{i=\mathrm{SeqInt}_\mathbb{Q} a}^\infty$ is a sequence of rationals and for each $i \geq \mathrm{SeqInt}_\mathbb{Q}\, a$, we have $c_i = (a_i)^{-1}$.*

Remark 5.3.17. This allows to denote c by a^{-1}. Axiom of substitution obeyed.

Remark 5.3.18. (i) If a, b are functions such that there exists an $m \in \mathbb{Z}$ so that $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ are sequences of rationals, then we set $a - b := a + (-b)$.

(ii) If a, b are functions such that there exists an $m \in \mathbb{Z}$ so that $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ are sequences of rationals and for each $i \geq m$, we have $b_i \neq 0_{\mathbb{Q}}$, then we set $a/b := ab^{-1}$.

Proposition 5.3.19 (Sums, products and negations of Cauchy sequences of rationals are Cauchy)**.** *Let a, b be functions and $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ are Cauchy sequences of rationals. Then $((a+b)_i)_{i=m}^{\infty}$, $((ab)_i)_{i=m}^{\infty}$, $((-a)_i)_{i=m}^{\infty}$ are Cauchy sequences of rationals.*

Definition 5.3.20 (Sequences of rationals bounded away from zero)**.** For any objects a, m, we write "$(a_i)_{i=m}^{\infty}$ is a sequence of rationals bounded away from zero" iff $(a_i)_{i=m}^{\infty}$ is a sequence of rationals and there exists a $c > 0_{\mathbb{Q}}$ such that for every $i \geq m$, we have $|a|_i \geq c$.

Remark 5.3.21. Axiom of substitution obeyed.

Remark 5.3.22. We'll abbreviate "$(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of rationals and $(a_i)_{i=m}^{\infty}$ is a sequence of rationals bounded away from zero" by "$(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of rationals bounded away from zero".

Proposition 5.3.23 (Reciprocation of Cauchy sequences of rationals bounded away from zero are Cauchy)**.** *Let a be a function and $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of rationals bounded away from zero. Then $a_i \neq 0_{\mathbb{Q}}$ for any $i \geq m$, and $((a^{-1})_i)_{i=m}^{\infty}$ is a Cauchy sequence of rationals bounded away from zero.*

Lemma 5.3.24 (Addition on $\mathbb{R}$)**.** *Let $x, y \in \mathbb{R}$. Then there exists a unique $z \in \mathbb{R}$ such that there exist functions a, b such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $((a+b)_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are Cauchy sequences of rationals, and $x = \mathrm{LIM}_{n\to\infty} a_n$, and $y = \mathrm{LIM}_{n\to\infty} b_n$ and $z = \mathrm{LIM}_{n\to\infty} (a+b)_n$.*

Remark 5.3.25. This allows to denote z by $x + y$. Axiom of substitution obeyed.

Lemma 5.3.26 (Multiplication on $\mathbb{R}$)**.** *Let $x, y \in \mathbb{R}$. Then there exists a unique $z \in \mathbb{R}$ such that there exist functions a, b such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $((ab)_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are Cauchy sequences of rationals, and $x = \mathrm{LIM}_{n\to\infty} a_n$, and $y = \mathrm{LIM}_{n\to\infty} b_n$ and $z = \mathrm{LIM}_{n\to\infty} (ab)_n$.*

Remark 5.3.27. This allows to denote z by xy. Axiom of substitution obeyed.

Lemma 5.3.28 (Negation on $\mathbb{R}$)**.** *Let $x \in \mathbb{R}$. Then there exists a unique $y \in \mathbb{R}$ such that there exists a function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $((-a)_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are Cauchy sequences of rationals, and $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} (-a)_n$.*

Remark 5.3.29. This allows to denote y by $-x$. Axiom of substitution obeyed.

Lemma 5.3.30 (The constant sequences of $1_{\mathbb{Q}}$'s and $0_{\mathbb{Q}}$'s)**.** *There exist unique functions a, b such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are sequences of rationals such that for each $i \geq 1_{\mathbb{Z}}$, we have $a_i = 0_{\mathbb{Q}}$ and $b_i = 1_{\mathbb{Q}}$. Further, any such a, b are Cauchy sequences of rationals.*

Remark 5.3.31. This allows to denote a and b by $\mathfrak{o}$ and $\mathfrak{1}$ respectively.

Lemma 5.3.32 (Characterizing non-zero reals)**.** *Let $x \in \mathbb{R}$. Then $x \neq \mathrm{LIM}_{n\to\infty} \mathfrak{o}_n \iff$ there exists a function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals bounded away from zero and $x = \mathrm{LIM}_{n\to\infty} a_n$.*

Lemma 5.3.33 (Reciprocation on $\mathbb{R}$)**.** *Let $x \in \mathbb{R}$ such that $x \neq \mathrm{LIM}_{n\to\infty} \mathfrak{o}_n$. Then there exists a unique $y \in \mathbb{R}$ such that there exists a function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals bounded away from zero, and $((a^{-1})_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals, and $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} (a^{-1})_n$.*

Remark 5.3.34. This allows to denote y by x^{-1}. Axiom of substitution obeyed.

Corollary 5.3.35. *Let a, b be functions such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are Cauchy sequences of rationals. Then $((a+b)_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $((ab)_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $((-a)_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are Cauchy sequences of rationals with*

(i) $(\mathrm{LIM}_{n\to\infty} a_n) + (\mathrm{LIM}_{n\to\infty} b_n) = \mathrm{LIM}_{n\to\infty} (a+b)_n$,

(ii) $(\mathrm{LIM}_{n\to\infty} a_n)(\mathrm{LIM}_{n\to\infty} b_n) = \mathrm{LIM}_{n\to\infty} (ab)_n$,

(iii) $-(\mathrm{LIM}_{n\to\infty} a_n) = \mathrm{LIM}_{n\to\infty} (-a)_n$, *and*

(iv) $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *is a sequence of rationals bounded away from zero* $\implies$ $((a^{-1})_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *is a Cauchy sequence of rationals, and* $\mathrm{LIM}_{n\to\infty} a_n \neq \mathrm{LIM}_{n\to\infty} \mathfrak{o}_n$ *and* $(\mathrm{LIM}_{n\to\infty} a_n)^{-1} = \mathrm{LIM}_{n\to\infty} (a^{-1})_n$

Proposition 5.3.36 ($\mathbb{R}$ forms a field)**.** *Let* $x, y, z \in \mathbb{R}$*. Then*

$$\begin{aligned} x + y &= y + x, \\ (x+y)+z &= x+(y+z), \\ x + \left(\operatorname*{LIM}_{n\to\infty} \mathfrak{o}_n\right) &= x, \\ x + (-x) &= \operatorname*{LIM}_{n\to\infty} \mathfrak{o}_n, \\ xy &= yx, \\ (xy)z &= x(yz), \\ x\left(\operatorname*{LIM}_{n\to\infty} \mathfrak{1}_n\right) &= x, \\ xx^{-1} &= \operatorname*{LIM}_{n\to\infty} \mathfrak{1}_n \qquad \text{if } x \neq \operatorname*{LIM}_{n\to\infty} \mathfrak{o}_n, \text{ and} \\ x(y+z) &= xy + xz. \end{aligned}$$

Remark 5.3.37. For any $x, y \in \mathbb{R}$, we set $x - y :\!= x + (-y)$. Further, if $y \neq \mathrm{LIM}_{n\to\infty} \mathfrak{o}_n$, we set $x/y :\!= xy^{-1}$.

Lemma 5.3.38. *Analogue of Lemma 4.2.23 holds.*

Proposition 5.3.39 ($\mathrm{LIM}_{n\to\infty} 1/n = \mathrm{LIM}_{n\to\infty} 0$)**.** *Let* a *be the function for which Proposition 5.1.5 holds. Then* $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} \mathfrak{o}_n$*.*

5.4 Ordering the reals

July 1, 2021

Definition 5.4.1 (Sequences of rationals positively and negatively bounded away from zero)**.** For any objects a, m, we write

(i) "$(a_i)_{i=m}^{\infty}$ is a sequence of rationals positively bounded away from zero" iff $(a_i)_{i=m}^{\infty}$ is a sequence of rationals and there exists a $c > 0_{\mathbb{Q}}$ such that for each $i \geq m$, we have $a_i \geq c$, and

(ii) "$(a_i)_{i=m}^{\infty}$ is a sequence of rationals negatively bounded away from zero" iff $(a_i)_{i=m}^{\infty}$ is a sequence of rationals and there exists a $c < 0_{\mathbb{Q}}$ such that for each $i \geq m$, we have $a_i \leq c$.

Remark 5.4.2. Axiom of substitution obeyed by both.

Remark 5.4.3. We'll abbreviate "$(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of rationals and $(a_i)_{i=m}^{\infty}$ is a sequence of rationals positively bounded away from zero" by $(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of rationals positively bounded away from zero, etc.

Definition 5.4.4 (Positive and negative reals)**.**

(i) An object x is called a positive real iff there exists an object a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals positively bounded away from zero and $x = \mathrm{LIM}_{n\to\infty} a_n$.

(ii) An object x is called a negative real iff there exists an object a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals negatively bounded away from zero and $x = \mathrm{LIM}_{n\to\infty} a_n$.

Remark 5.4.5. Axiom of substitution obeyed.

Corollary 5.4.6 (Characterization of positive and negative reals)**.** *The analogue of Corollary 4.1.41 holds.*

Proposition 5.4.7 (Sums and products of positives are positive)**.** *Let x, y be positive reals. Then $x + y$ and xy are positive reals.*

Proposition 5.4.8 (Trichotomy for $\mathbb{R}$)**.** *Let $x \in \mathbb{R}$. Then exactly one of the following holds:*

(i) x is a positive real.
(ii) $x = \mathrm{LIM}_{n\to\infty} \mathfrak{o}_n$.
(iii) x is negative real.

Definition 5.4.9 (Order on $\mathbb{R}$)**.** For any objects x, y, we write

(i) "$x > y$", or "$y < x$", iff $x, y \in \mathbb{R}$ and $x - y$ is a positive real, and
(ii) "$x \geq y$", or "$y \leq x$", iff $x > y$ or ($x \in \mathbb{R}$ and $x = y$).

Remark 5.4.10. Axiom of substitution obeyed. Different symbol should have been used.

Corollary 5.4.11 (Characterizing " $>$ " and " $\geq$ ", and order properties for $\mathbb{R}$)**.** *The analogues of Corollaries 4.2.31 and 4.2.32, and Proposition 4.2.33 hold.*

Lemma 5.4.12 (Constant rational sequences are Cauchy). *Let $r \in \mathbb{Q}$. Then there exists a unique function a such that $(a_i)_{i=1_\mathbb{Z}}^\infty$ is a sequence of rationals and $a_i = r$ for each $i \geq 1_\mathbb{Z}$. Further, for such an a, we have that $(a_i)_{i=1_\mathbb{Z}}^\infty$ is a Cauchy sequence of rationals.*

Remark 5.4.13. This allows to denote a by $\operatorname{Seq}_\mathbb{Q} r$. Axiom of substitution obeyed.

Lemma 5.4.14 (Embedding $\mathbb{Q}$ into $\mathbb{R}$). *There exists a unique function f such that $f : \mathbb{Q} \to \mathbb{R}$ such that $f(r) = \operatorname{LIM}_{n\to\infty} (\operatorname{Seq}_\mathbb{Q} r)_n$ for each $r \in \mathbb{Q}$. Further, for such an f, for any $r, s \in \mathbb{Q}$, we have*

(i) $r = s \iff f(r) = f(s)$,
(ii) $f(r + s) = f(r) + f(s)$, *and*
(iii) $f(rs) = f(r)f(s)$.

Remark 5.4.15. This allows to set $r_{\mathbb{R}''} := f(r)$ for each $r \in \mathbb{Q}$. Also, we set $p_{\mathbb{R}'} := (p_{\mathbb{Q}'})_{\mathbb{R}''}$ for each $p \in \mathbb{Z}$ and $n_\mathbb{R} := (n_\mathbb{Q})_{\mathbb{R}''}$ for each $n \in \mathbb{N}$.

Corollary 5.4.16 (Properties of embedding). *Let $r, s \in \mathbb{Q}$. Then*

(i) $(-r)_{\mathbb{R}''} = -r_{\mathbb{R}''}$,
(ii) $(r - s)_{\mathbb{R}''} = r_{\mathbb{R}''} - s_{\mathbb{R}''}$,
(iii) $r \neq 0_\mathbb{Q} \implies r_{\mathbb{R}''} \neq \operatorname{LIM}_{n\to\infty} \mathsf{o}_n$ *and* $(r^{-1})_{\mathbb{R}''} = (r_{\mathbb{R}''})^{-1}$,
(iv) $s \neq 0_\mathbb{Q} \implies s_{\mathbb{R}''} \neq \operatorname{LIM}_{n\to\infty} \mathsf{o}_n$ *and* $(r/s)_{\mathbb{R}''} = r_{\mathbb{R}''}/s_{\mathbb{R}''}$,
(v) r *is a positive rational* $\iff r_{\mathbb{R}''}$ *is a positive real,*
(vi) r *is a negative rational* $\iff r_{\mathbb{R}''}$ *is a negative real*
(vii) $r > s \iff r_{\mathbb{R}''} > s_{\mathbb{R}''}$, *and*
(viii) $r \geq s \iff r_{\mathbb{R}''} \geq s_{\mathbb{R}''}$.

Corollary 5.4.17. *(i) Let $n \in \mathbb{N}$. Then $(n_\mathbb{Z})_{\mathbb{R}'} = n_\mathbb{R}$.*
(ii) Let $p, q \in \mathbb{Z}$ and $q \neq 0_\mathbb{Z}$. Then $q_{\mathbb{R}'} \neq 0_\mathbb{R}$, and $q_{\mathbb{Q}'} \neq 0_\mathbb{Q}$ and $p_{\mathbb{R}'}/q_{\mathbb{R}'} = (p_{\mathbb{Q}'}/q_{\mathbb{Q}'})_{\mathbb{R}''}$.

Remark 5.4.18. We still need to develop limits for reals and discard the $\operatorname{LIM}_{n\to\infty}\ {}_n$ scaffolding. Also, when we develop exponentiation, absolute values, sequences, etc. for reals, we'll need to again verify this embedding.

Lemma 5.4.19 (Absolute value on $\mathbb{R}$, and exponentiation of reals by naturals and integers). *Analogues of all the results about absolute value and exponentiation, from Lemma 4.3.3 to Corollary 4.3.25 hold.*

Lemma 5.4.20 (Embedding is consistent with absolute value and exponentiation). *Let $r, s \in \mathbb{Q}$, and $n \in \mathbb{N}$ and $p \in \mathbb{Z}$. Then*

(i) $|r|_{\mathbb{R}''} = |r_{\mathbb{R}''}|$,
(ii) $d(r, s)_{\mathbb{R}''} = d(r_{\mathbb{R}''}, s_{\mathbb{R}''})$,
(iii) $(r^n)_{\mathbb{R}''} = (r_{\mathbb{R}''})^n$, *and*
(iv) $r \neq 0_{\mathbb{Q}} \implies r_{\mathbb{R}''} \neq 0_{\mathbb{R}}$ *and* $(r^p)_{\mathbb{R}''} = (r_{\mathbb{R}''})^p$.

Lemma 5.4.21 (Cauchy sequences of non-negative rationals form non-negative reals). *Let a be a function such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals such that $a_i \geq 0_{\mathbb{Q}}$ for each $i \geq 1_{\mathbb{Z}}$. Then $\mathrm{LIM}_{n\to\infty} a_n \geq 0_{\mathbb{R}}$.*

Corollary 5.4.22. *Let a, b be functions such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are Cauchy sequences of rationals and there exists an $N \geq 1_{\mathbb{Z}}$ such that $a_i \geq b_i$ for each $i \geq N$. Then $\mathrm{LIM}_{n\to\infty} a_n \geq \mathrm{LIM}_{n\to\infty} b_n$.*

Lemma 5.4.23 (A positive real is greater than a positive rational). *Let $x > 0_{\mathbb{R}}$. Then there exists an $r > 0_{\mathbb{Q}}$ such that $x > r_{\mathbb{R}''}$.*

Proposition 5.4.24 (Rationals between reals). *Let $x, y \in \mathbb{R}$ such that $x < y$. Then there exists an $r \in \mathbb{Q}$ such that $x < r_{\mathbb{R}''} < y$.*

Lemma 5.4.25 (Reals surrounded by rationals). *Let $x \in \mathbb{R}$ and $\varepsilon > 0_{\mathbb{Q}}$. Then there exist $r, s \in \mathbb{Q}$ such that $r_{\mathbb{R}''} < x < s_{\mathbb{R}''}$ and $s - r < \varepsilon$.*

Proposition 5.4.26. *Analogues of results from Lemma 4.4.1 to Corollary 4.4.7 hold.*

Remark 5.4.27. The analogue of Corollary 4.4.4 (iii) allows to denote p by $\lfloor x \rfloor$ for the case of reals. Axiom of substitution obeyed.

Lemma 5.4.28 (Embedding is consistent with floor). *Let $r \in \mathbb{Q}$. Then $\lfloor r \rfloor = \lfloor r_{\mathbb{R}''} \rfloor$.*

Proposition 5.4.29. *Let $x \in \mathbb{R}$ and a be a function such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals. Then*

(i) there exists an $N \geq 1_{\mathbb{Z}}$ such that $x \geq (a_i)_{\mathbb{R}''}$ for each $i \geq N$ $\implies$ $x \geq \mathrm{LIM}_{n\to\infty} a_n$, and
(ii) there exists an $N \geq 1_{\mathbb{Z}}$ such that $x \leq (a_i)_{\mathbb{R}''}$ for each $i \geq N$ $\implies$ $x \leq \mathrm{LIM}_{n\to\infty} a_n$.

5.5 The least upper bound property

July 11, 2021

Definition 5.5.1 (Upper bounds and least upper bounds)**.** For any objects M, E, we write

(i) "M is an upper bound for E" iff $M \in \mathbb{R}$, and $E \subseteq \mathbb{R}$ and $x \leq M$ for each $x \in E$, and

(ii) "M is a least upper bound for E" iff M is an upper bound for E and for any object M', we have that M' is an upper bound for $E \implies M' \geq M$.

Remark 5.5.2. Axiom of substitution obeyed by both.

Remark 5.5.3. From now on, we may omit (only) the redundant specification of object types, which might be implied by the relations that follow.

Lemma 5.5.4. *Let $M, M' \in \mathbb{R}$ and $E \subseteq \mathbb{R}$ such that M is an upper bound for E and $M' \geq M$. Then M' is a least upper bound for E.*

Lemma 5.5.5 (At most one least upper bound)**.** *Let $E \subseteq \mathbb{R}$. Then there exists at most one $M \in \mathbb{R}$ such that M is an upper bound for E.*

Proposition 5.5.6. *(i) Let $M \in \mathbb{R}$. Then M is an upper bound for $\emptyset$.*

(ii) There is no $M \in \mathbb{R}$ such that M is the least upper bound for $\emptyset$.

(iii) There is no $M \in \mathbb{R}$ such that M is an upper bound for $\{x \in \mathbb{R} : x > 0_{\mathbb{R}}\}$.

(iv) $1_{\mathbb{R}}$ is a least upper bound for $\{x \in \mathbb{R} : 0_{\mathbb{R}} \leq x \leq 1_{\mathbb{R}}\}$.

Lemma 5.5.7 (Creating a sequence of rational upper bounds for an upper bounded set)**.** *Let $E \subseteq \mathbb{R}$, and $n \geq 1_{\mathbb{Z}}$ and $K, L \in \mathbb{Z}$ such that $K_{\mathbb{R}'}/n_{\mathbb{R}'}$ is not an upper bound for E, but $L_{\mathbb{R}'}/n_{\mathbb{R}'}$ is an upper bound for E. Then there exists a unique $m \in \mathbb{Z}$ such that $(m - 1_{\mathbb{Z}})_{\mathbb{R}'}/n_{\mathbb{R}'}$ is not an upper bound for E, but $m_{\mathbb{R}'}/n_{\mathbb{R}'}$ is an upper bound for E.*

Theorem 5.5.8 (Least upper bound property of $\mathbb{R}$)**.** *Let $E \subseteq \mathbb{R}$ such that $E \neq \emptyset$ and there exists an $M \in \mathbb{R}$ such that M is an upper bound for E. Then there exists a unique $S \in \mathbb{R}$ such that S is a least upper bound for E.*

Definition 5.5.9 (Lower bounds and greatest lower bounds)**.** For any objects M, E, we write

(i) "M is a lower bound for E" iff $M \in \mathbb{R}$, and $E \subseteq \mathbb{R}$ and $x \geq M$ for each $x \in E$, and
(ii) "M is a greatest lower bound for E" iff M is a lower bound for E and for any object M', we have that M' is a lower bound for $E \implies M' \leq M$.

Remark 5.5.10. For any set $E \subseteq \mathbb{R}$, we set $-E \coloneqq \{-x : x \in E\}$.

Lemma 5.5.11. *Let $E \subseteq \mathbb{R}$. Then $-E \subseteq \mathbb{R}$ and $-(-E) = E$.*

Lemma 5.5.12 (Relating upper and lower bounds)**.** *Let $M \in \mathbb{R}$ and $E \subseteq \mathbb{R}$. Then*

(i) (a) M is an upper bound for $E \iff -M$ is a lower bound for $-E$,
(b) M is a lower bound for $E \iff -M$ is an upper bound for $-E$,
(ii) (a) M is a least upper bound for $E \iff -M$ is a greatest lower bound for $-E$, and
(b) M is a greatest lower bound for $E \iff -M$ is a least upper bound for $-E$.

Corollary 5.5.13 (Greatest lower bound property for $\mathbb{R}$)**.** *Let $E \subseteq \mathbb{R}$ such that $E \neq \emptyset$ and there exists an $M \in \mathbb{R}$ such that M is a lower bound for E. Then there exists a unique $S \in \mathbb{R}$ such that S is a greatest lower bound for E.*

Proposition 5.5.14 (Square root of 2)**.** *There exists a unique $x \geq 0_{\mathbb{R}}$ such that $x^2 = 2_{\mathbb{R}}$.*

Remark 5.5.15. The proof of the above uses just the least upper bound property, in addition to ordered field axioms. Therefore, $\mathbb{Q}$ does not obey least upper bound property.

Definition 5.5.16 (Irrationals)**.** An object x is called an irrational iff $x \in \mathbb{R}$ and there does not exist any $r \in \mathbb{Q}$ such that $x = r_{\mathbb{R}''}$.

Remark 5.5.17. Axiom of substitution obeyed.

Lemma 5.5.18 (There are irrationals between reals)**.** *Let $x, y \in \mathbb{R}$ such that $x < y$. Then there exists an irrational z such that $x < z < y$.*

5.6 Real exponentiation, part I

July 13, 2021

Lemma 5.6.1 (n-th roots of non-negative reals)**.** *Let $x \geq 0_{\mathbb{R}}$ and $n \geq 1$. Then there exists a unique $z \in \mathbb{R}$ such that z is the least upper bound for $\{y \in \mathbb{R} : y^n \leq x\}$.*

Remark 5.6.2. This allows to denote x by $x^{1/n}$. Axiom of substitution obeyed.

Proposition 5.6.3. *Let $n \geq 1$. Then*

(i) $(0_{\mathbb{R}})^{1/n} = 0_{\mathbb{R}}$, *and*
(ii) $(1_{\mathbb{R}})^{1/n} = 1_{\mathbb{R}}$.

Lemma 5.6.4. *Let $x \geq 0_{\mathbb{R}}$, and $n \in \mathbb{N}$ and $\varepsilon \geq 0_{\mathbb{R}}$. Then*

(i) $\varepsilon \leq 1_{\mathbb{R}} \implies (x+\varepsilon)^n \leq x^n + ((1_{\mathbb{R}}+x)^n - x^n)\varepsilon$, *and*
(ii) $\varepsilon \leq x \implies (x-\varepsilon)^n \geq x^n - ((1_{\mathbb{R}}+x)^n - x^n)\varepsilon$.

Lemma 5.6.5. *Let $x, y \geq 0_{\mathbb{R}}$ and $n \in \mathbb{N}$. Then*

(i) $x^n < y \implies (x+\varepsilon)^n < y$ *for some* $0_{\mathbb{R}} < \varepsilon < 1_{\mathbb{R}}$, *and*
(ii) $x^n > y$ *and* $x > 0_{\mathbb{R}} \implies (x-\varepsilon)^n > y$ *for some* $0_{\mathbb{R}} < \varepsilon < x$.

Proposition 5.6.6 (n-th power cancels the n-th root)**.** *Let $x \geq 0_{\mathbb{R}}$ and $n \geq 1$. Then $(x^{1/n})^n = x$. In particular, $x^{1/1} = x$.*

Corollary 5.6.7. *Let $x, y \geq 0_{\mathbb{R}}$ and $n \geq 1$. Then*

(i) $x^{1/n} \geq 0_{\mathbb{R}}$,
(ii) $x^n \geq 0_{\mathbb{R}}$ *and* $(x^n)^{1/n} = x$, *and*
(iii) $y^n = x \implies y = x^{1/n}$.

Proposition 5.6.8 (Properties of n-th roots of non-negative reals)**.** *Let $x, y \geq 0_{\mathbb{R}}$ and $z > 0_{\mathbb{R}}$. Let $m, n \geq 1$ and $k \in \mathbb{N}$. Let $p \in Z$. Then $xy, x^{1/m}, z^{-1}, |x|, x^k, z^p \geq 0_{\mathbb{R}}$, and $z^{1/n} \neq 0_{\mathbb{R}}$ and $mn \geq 1$, and*

(i) $(xy)^{1/n} = x^{1/n}y^{1/n}$,
(ii) $x^{1/(mn)} = (x^{1/m})^{1/n}$,
(iii) $(z^{-1})^{1/n} = (z^{1/n})^{-1}$,
(iv) *(a)* $x = 0_{\mathbb{R}} \iff x^{1/n} = 0_{\mathbb{R}}$,
(b) *x is a positive real* $\iff$ *$x^{1/n}$ is a positive real,*
(v) $x > y > 0_{\mathbb{R}} \implies x^{1/n} > y^{1/n} > 0_{\mathbb{R}}$,

(vi) $|x|^{1/n} = \left|x^{1/n}\right| = x^{1/n}$,
(vii) $(x^k)^{1/m} = (x^{1/m})^k$, *and*
(viii) $(z^p)^{1/n} = (z^{1/n})^p$.

Corollary 5.6.9.
(i) *Let* $x, y \geq 0_{\mathbb{R}}$ *and* $n \geq 1$ *such that* $x^{1/n} = y^{1/n}$. *Then* $x = y$.
(ii) *Let* $x \in \mathbb{R}$ *and* $1 \leq m < n$. *Then*
  (a) $0_{\mathbb{R}} < x < 1_{\mathbb{R}} \implies x^{1/n} > x^{1/m}$, *and*
  (b) $x > 1_{\mathbb{R}} \implies x^{1/m} > x^{1/n}$.

Lemma 5.6.10 (Exponentiation of positive reals by rationals)**.** *Let* $x > 0_{\mathbb{R}}$ *and* $r \in \mathbb{Q}$. *Then there exists a unique* $y \in \mathbb{R}$ *such that there exist an* $m \in \mathbb{Z}$ *and an* $n \geq 1$ *such that* $r = m_{\mathbb{Q}'}/n_{\mathbb{Q}}$, *and* $x^{1/n} \neq 0_{\mathbb{R}}$ *and* $y = (x^{1/n})^m$.

Remark 5.6.11. This allows to denote y by x^r. Axiom of substitution obeyed.

Corollary 5.6.12. *Let* $x > 0_{\mathbb{R}}$ *and* $r \in \mathbb{Q}$. *Then* $x^r > 0_{\mathbb{R}}$.

Corollary 5.6.13. *Let* $x > 0_{\mathbb{R}}$, *and* $n \in \mathbb{N}$ *and* $m \in \mathbb{Z}$. *Then*
(i) $n \geq 1 \implies x^{1/n} \neq 0_{\mathbb{R}}$ *and* $x^{m_{\mathbb{Q}'}/n_{\mathbb{Q}}} = (x^{1/n})^m$,
(ii) $x^{n_{\mathbb{Q}}} = x^n$,
(iii) $x \neq 0_{\mathbb{Q}}$ *and* $x^{m_{\mathbb{Q}'}} = x^m$, *and in particular,* $x^{-1_{\mathbb{Q}}} = x^{-1}$, *and*
(iv) $n \geq 1 \implies x^{1_{\mathbb{Q}}/n_{\mathbb{Q}}} = x^{1/n}$.

Corollary 5.6.14. *Let* $x > 0_{\mathbb{R}}$. *Let* $m, n \in \mathbb{N}$ *such that* $n \geq 1$. *Let* $p \in \mathbb{Z}$. *Then* $x^{1/n} \neq 0_{\mathbb{R}}$ *and*
(i) $(x^{1/n})^m = x^{m_{\mathbb{Q}'}/n_{\mathbb{Q}}}$,
(ii) $x^m = x^{m_{\mathbb{Q}}}$,
(iii) $x^p = x^{p_{\mathbb{Q}'}}$, *and in particular,* $x^{-1_{\mathbb{Q}}} = x^{-1}$, *and*
(iv) $x^{1/n} = x^{1_{\mathbb{Q}}/n_{\mathbb{Q}}}$.

Proposition 5.6.15. *Let* $r \in \mathbb{Q}$. *Then* $(1_{\mathbb{R}})^r = 1_{\mathbb{R}}$.

Proposition 5.6.16 (Properties of exponentiation of positive reals by rationals)**.** *Let* $x, y > 0_{\mathbb{R}}$ *and* $r, s \in \mathbb{Q}$. *Let* $n, m \in \mathbb{N}$ *such that* $m \geq 1$. *Let* $p \in \mathbb{Z}$. *Then* $xy, x^r, x^{-1}, |x|, x^n, x^{1/m} > 0_{\mathbb{R}}$ *and*
(i) $(xy)^r = x^r y^r$,
(ii) $x^{r+s} = x^r x^s$,

(iii) $x^{rs} = (x^r)^s$,
(iv) $x^{-1} > 0_{\mathbb{R}}$, *and* $x^r \neq 0_{\mathbb{R}}$ *and* $(x^{-1})^r = (x^r)^{-1} = x^{-r}$,
(v) x^r *is a positive real,*
(vi) $r > 0_{\mathbb{Q}}$ *and* $x > y > 0_{\mathbb{R}} \implies x^r > y^r > 0_{\mathbb{R}}$,
(vii) $|x|^r = |x^r| = x^r$,
(viii) $(x^n)^r = (x^r)^n = x^{n_{\mathbb{Q}} r}$,
(ix) $(x^p)^r = (x^r)^p = x^{p_{\mathbb{Q}'} r}$, *and*
(x) $(x^{1/n})^r = (x^r)^{1/n} = x^{r/n_{\mathbb{Q}}}$.

Corollary 5.6.17.

(i) *Let* $x, y > 0_{\mathbb{R}}$ *and* $q \in \mathbb{Q}$ *such that* $q \neq 0$ *and* $x^q = y^q$. *Then* $x = y$.
(ii) *Let* $x \in \mathbb{R}$ *and* $r, s \in \mathbb{Q}$ *such that* $r < s$. *Then*
  (a) $0_{\mathbb{R}} < x < 1_{\mathbb{R}} \implies x^r > x^s$, *and*
  (b) $x > 1_{\mathbb{R}} \implies x^s > x^r$.

Chapter 6

Limits of sequences

6.1 Convergence and limit laws

July 15, 2021

Definition 6.1.1 (Sequences of reals)**.** (i) For any objects a, m, n, we write "$(a_i)_{i=m}^n$ is a sequence of reals" iff $m, n \in \mathbb{Z}$ and $a\colon \{i \in \mathbb{Z} : m \leq i \leq n\} \to \mathbb{R}$.

(ii) For any objects a, m, we write "$(a_i)_{i=m}^\infty$ is a sequence of reals" iff $m \in \mathbb{Z}$ and $a\colon \{i \in \mathbb{Z} : i \geq m\} \to \mathbb{R}$.

Remark 6.1.2. Axiom of substitution obeyed by both.

Lemma 6.1.3 (Beginning integers of real sequences)**.** *Let a be a function such that there exists an $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^\infty$ is a sequence of reals. Then there exists a unique $n \in \mathbb{Z}$ such that $(a_i)_{i=n}^\infty$ is a sequence of reals.*

Remark 6.1.4. This allows to denote n by $\operatorname{SeqInt}_{\mathbb{R}} a$. Axiom of substitution obeyed.

Lemma 6.1.5. *Let a be a function such that there exists an $m \in \mathbb{Z}$ so that $(a_i)_{i=m}^\infty$ is a sequence of rationals. Then there exists a unique function b such that $(b_i)_{i=\operatorname{SeqInt}_{\mathbb{Q}} a}^\infty$ is a sequence of reals such that $b_i = (a_i)_{\mathbb{R}''}$ for each $i \geq \operatorname{SeqInt}_{\mathbb{Q}} a$. Further, for such a function b, we have that $\operatorname{SeqInt}_{\mathbb{R}} b = \operatorname{SeqInt}_{\mathbb{Q}} a$.*

Remark 6.1.6. This allows to denote b by $a_{\mathbb{R}''}$. Axiom of substitution obeyed.

Definition 6.1.7 (Cauchy sequences of reals)**.** For any objects a, m, we write "$(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of reals" iff $(a_i)_{i=m}^{\infty}$ is a sequence of reals and for every $\varepsilon > 0_{\mathbb{R}}$, there exists an $N \geq m$ such that for all $i, j \geq N$, we have $|a_i - a_j| \leq \varepsilon$.

Remark 6.1.8. Axiom of substitution obeyed.

Lemma 6.1.9 (Embedding consistent for Cauchy sequences)**.** *Let $(a_i)_{i=m}^{\infty}$ be a sequence of rationals. Then $(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of rationals $\iff$ $((a_{\mathbb{R}''})_i)_{i=m}^{\infty}$ is a Cauchy sequence of reals.*

Lemma 6.1.10 (Characterization of Cauchy sequences of reals using rational ε's)**.** *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals. Then $(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of reals $\iff$ for every $\varepsilon > 0_{\mathbb{Q}}$, there exists an $N \geq m$ such that for each $i, j \geq N$, we have $|a_i - a_j| \leq \varepsilon_{\mathbb{R}''}$.*

Definition 6.1.11 (Sequences converging in $\mathbb{R}$)**.** For any objects a, m, L, we write "$(a_i)_{i=m}^{\infty}$ converges to L in $\mathbb{R}$" iff $(a_i)_{i=m}^{\infty}$ is a sequence of reals, and $L \in \mathbb{R}$ and for each $\varepsilon > 0_{\mathbb{R}}$, there exists an $N \geq m$ such that for each $i \geq N$, we have $|a_i - L| \leq \varepsilon$.

Remark 6.1.12. Axiom of substitution obeyed.

Lemma 6.1.13 (Characterization of sequences converging in $\mathbb{R}$ using rational ε's)**.** *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals and $L \in \mathbb{R}$. Then $(a_i)_{i=m}^{\infty}$ converges to L in $\mathbb{R}$ $\iff$ for each $\varepsilon > 0_{\mathbb{Q}}$, there exists an $N \geq m$ such that for each $i \geq N$, we have $|a_i - L| \leq \varepsilon_{\mathbb{R}''}$.*

Proposition 6.1.14. *Let $x \in \mathbb{R}$ such that $|x| > 1_{\mathbb{R}}$. Then there exists a unique function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a sequence of reals so that for each $i \geq 1_{\mathbb{Z}}$, we have $x \neq 0_{\mathbb{R}}$ and $a_i = x^{-i}$. Further, for such an a, we have that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ converges to $0_{\mathbb{R}}$ in $\mathbb{R}$.*

Definition 6.1.15 (Convergent sequences of reals)**.** For any objects a, m, we write "$(a_i)_{i=m}^{\infty}$ is a convergent sequence of reals" iff there exists an object L such that $(a_i)_{i=m}^{\infty}$ converges to L in $\mathbb{R}$.

Remark 6.1.16. Axiom of substitution obeyed.

Proposition 6.1.17 (Limits of convergent sequences)**.** *Let a be a function such that there exists an $m \in \mathbb{Z}$ so that $(a_i)_{i=m}^{\infty}$ be a convergent sequence of reals. Then there exists a unique $L \in \mathbb{R}$ such that $(a_i)_{i=\mathrm{SeqInt}_{\mathbb{R}}\, a}^{\infty}$ converges to L in $\mathbb{R}$.*

Remark 6.1.18. This allows to denote L by $\lim_{n\to\infty} a_n$. Axiom of substitution obeyed.

Proposition 6.1.19 (Convergent sequences are Cauchy)**.** *Let $(a_i)_{i=m}^{\infty}$ be a convergent sequence of reals. Then $(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of reals.*

Proposition 6.1.20 (Getting rid of "$\text{LIM}_{n\to\infty\ n}$")**.** *Let $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ be a Cauchy sequence of rationals. Then $((a_{\mathbb{R}''})_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a convergent sequence of reals with $\lim_{n\to\infty} (a_{\mathbb{R}''})_n = \text{LIM}_{n\to\infty} a_n$.*

Corollary 6.1.21 ($\lim_{n\to\infty} 1/n = 0$)**.** *There exists a unique function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a sequence of reals so that for each $i \geq 1_{\mathbb{Z}}$, we have $i_{\mathbb{R}'} \neq 0_{\mathbb{R}}$ and $a_i = (i_{\mathbb{R}'})^{-1}$. Further, for any such a, we have that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ converges to $0_{\mathbb{R}}$ in $\mathbb{R}$.*

Proposition 6.1.22 (Limits are independent of finite initial conditions)**.** *Let $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=n}^{\infty}$ be convergent sequences of reals and let one of the following hold:*

- *(i) There exists an $N \geq m, n$ such that for each $i \geq N$, we have $a_i = b_i$.*
- *(ii) There exists a $k \geq n - m$ such that for each $i \geq m$, we have $i + k \geq n$ and $a_i = b_{i+k}$.*

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Definition 6.1.23 (Divergent sequences of reals)**.** For any objects a, m, we write "$(a_i)_{i=m}^{\infty}$ is a divergent sequence of reals" iff $(a_i)_{i=m}^{\infty}$ is a sequence of reals such that $(a_i)_{i=m}^{\infty}$ is not a convergent sequence of reals.

Remark 6.1.24. Axiom of substitution obeyed.

Definition 6.1.25 (Bounded sequences of reals)**.**

- (i) For any objects a, m, n, we write "$(a_i)_{i=m}^{n}$ is a bounded sequence of reals " iff $(a_i)_{i=m}^{n}$ is a sequence of reals and there exists an object M such that for each $m \leq i \leq n$, we have $|a_i| \leq M$.
- (ii) For any objects a, m, we write "$(a_i)_{i=m}^{\infty}$ is a bounded sequence of reals" iff $(a_i)_{i=m}^{\infty}$ is a sequence of reals and there exists an object M such that for each $i \geq m$, we have $|a_i| \leq M$.

Remark 6.1.26. Axiom of substitution obeyed.

Lemma 6.1.27 (Embedding consistent for bounded sequences)**.** *Let $(a_i)_{i=m}^{\infty}$ be a sequence of rationals. Then $(a_i)_{i=m}^{\infty}$ is a bounded sequence of rationals $\iff$ $((a_{\mathbb{R}''})_i)_{i=m}^{\infty}$ is a bounded sequence of rationals.*

Lemma 6.1.28 (Cauchy sequences of reals are bounded)**.** *Analogues of results from Lemma 5.1.8 to Lemma 5.1.10 hold.*

Corollary 6.1.29 (Convergent sequences are bounded)**.** *Let $(a_i)_{i=m}^{\infty}$ be a convergent sequence of reals. Then $(a_i)_{i=m}^{\infty}$ is a bounded sequence of reals.*

Definition 6.1.30 (Equivalent sequences of reals)**.** For any objects a, b, m, n, we write "$(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=n}^{\infty}$ are equivalent sequences of reals" iff $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=n}^{\infty}$ are sequences of reals, and $m = n$ and for each $\varepsilon > 0_{\mathbb{R}}$, there exists an $N \geq m, n$ such that for each $i \geq N$, we have $|a_i - b_i| \leq \varepsilon$.

Remark 6.1.31. Axiom of substitution obeyed.

Lemma 6.1.32 (Embedding consistent for equivalent sequences)**.** *Let $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=n}^{\infty}$ be sequences of rationals. Then $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=n}^{\infty}$ are equivalent sequences of rationals $\iff$ $((a_{\mathbb{R}''})_i)_{i=m}^{\infty}$ and $((b_{\mathbb{R}''})_i)_{i=n}^{\infty}$ are equivalent sequences of reals.*

Lemma 6.1.33 (Characterizing equivalent sequences of reals using rational ε's)**.** *Let $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=n}^{\infty}$ be sequences of reals. Then $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=n}^{\infty}$ are equivalent sequences of reals $\iff$ $m = n$ and for each $\varepsilon > 0_{\mathbb{Q}}$, there exists an $N \geq m, n$ such that for each $i \geq N$, we have $|a_i - b_i| \leq \varepsilon_{\mathbb{R}''}$.*

Proposition 6.1.34 (Properties of equivalent sequences of reals)**.** *Let $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=m}^{\infty}$ be equivalent sequences of reals and $L \in \mathbb{R}$. Then*

(i) the analogues of (i) and (ii) of Proposition 5.2.5 hold, and

(ii) $(a_i)_{i=m}^{\infty}$ converges to L in $\mathbb{R}$ $\iff$ $(b_i)_{i=m}^{\infty}$ converges to L in $\mathbb{R}$.

Lemma 6.1.35 (Sums, products, negations and reciprocations of real sequences)**.** *The analogues of results from Lemma 5.3.12 to Remark 5.3.18 hold.*

Lemma 6.1.36 (Embedding consistent for operations on sequences)**.** *(i) Let a, b be functions such that there exists an $m \in \mathbb{Z}$ so that $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ are sequences of rationals. Then there exists an $m \in \mathbb{Z}$ such that $((a_{\mathbb{R}''})_i)_{i=m}^{\infty}$, $((b_{\mathbb{R}''})_i)_{i=m}^{\infty}$ are sequences of reals, and*

(i) $(a + b)_{\mathbb{R}''} = a_{\mathbb{R}''} + b_{\mathbb{R}''}$,

(ii) $(ab)_{\mathbb{R}''} = a_{\mathbb{R}''} b_{\mathbb{R}''}$, and

(iii) $(-a)_{\mathbb{R}''} = -(a_{\mathbb{R}''})$.

Further, if a is such that there exists an $m \in \mathbb{Z}$ so that $(a_i)_{i=m}^{\infty}$ is a sequence of rationals and for each $i \geq m$, we have $a_i \neq 0_{\mathbb{R}}$, then there exists an $m \in \mathbb{Z}$ such that $((a_{\mathbb{R}''})_i)_{i=m}^{\infty}$ is a sequence of reals, and for each $i \geq m$, we have $(a_{\mathbb{R}''})_i \neq 0_{\mathbb{R}}$, and $(a^{-1})_{\mathbb{R}''} = (a_{\mathbb{R}''})^{-1}$.

Remark 6.1.37. It easily follows that the difference and quotient of sequences is also consistent in the above fashion.

Lemma 6.1.38 (min and max functions)**.** *There exist unique functions f, g such that $f, g \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so that for each $p \in \mathbb{R} \times \mathbb{R}$, exactly one of the following holds:*

(a) $p_1 \leq p_2$, *and* $f(p) = p_2$ *and* $g(p) = p_1$.
(b) $p_1 > p_2$, *and* $f(p) = p_1$ *and* $g(p) = p_2$.

Remark 6.1.39. This allows to denote f and g be max and min respectively.

Corollary 6.1.40 (Properties of min and max)**.** *Let $x, y \in \mathbb{R}$. Then*

(i) $x \leq y \implies \max((x,y)) = y$ *and* $\min((x,y)) = x$,
(ii) $x > y \implies \max((x,y)) = x$ *and* $\min((x,y)) = y$,
(iii) $\min((x,y)) \leq x, y \leq \max((x,y))$,
(iv) $\max((x,y)) = -\min((-x,-y))$, *and*
(v) $\min((x,y)) = -\max((-x,-y))$.

Lemma 6.1.41 (min and max operations on sequences)**.** *Let a, b be functions such that there exists an $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ are sequences of reals. Then $\operatorname{SeqInt}_{\mathbb{R}} a = \operatorname{SeqInt}_{\mathbb{R}} b$, and there exist unique functions c, d such that $(c_i)_{i=\operatorname{SeqInt}_{\mathbb{R}} a}^{\infty}$, $(d_i)_{i=\operatorname{SeqInt}_{\mathbb{R}} a}^{\infty}$ are sequences of reals and for each $i \geq \operatorname{SeqInt}_{\mathbb{R}} a$, we have $c_i = \max((a_i, b_i))$ and $d_i = \min((a_i, b_i))$.*

Remark 6.1.42. This allows to denote c and d by $\max((a,b))$ and $\min((a,b))$ respectively. Axiom of substitution obeyed.

Lemma 6.1.43. *Let $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ be sequences of reals. Then*

(i) $\max((a,b)) = -\min((-a,-b))$, *and*
(ii) $\min((a,b)) = -\max((-a,-b))$.

Lemma 6.1.44 (Absolute sequences of reals)**.** *Let a be a function such that there exists an $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^{\infty}$ is a sequence of reals. Then there exists a unique function c such that $(c_i)_{i=\operatorname{SeqInt}_{\mathbb{R}} a}^{\infty}$ is a sequence of reals so that for each $i \geq m$, we have $c_i = |a_i|$.*

Remark 6.1.45. This allows to denote c by $|a|$. Axiom of substitution obeyed.

Theorem 6.1.46 (Limit laws)**.** *Let $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=m}^{\infty}$ be convergent sequences of reals. Then $((a+b)_i)_{i=m}^{\infty}$, $((ab)_i)_{i=m}^{\infty}$, $((\min((a,b)))_i)_{i=m}^{\infty}$, $((-a)_i)_{i=m}^{\infty}$, $(|a|_i)_{i=m}^{\infty}$ are convergent sequences of reals with*

$$\begin{aligned}
\lim_{n\to\infty}(a+b)_n &= \lim_{n\to\infty}a_n + \lim_{n\to\infty}b_n,\\
\lim_{n\to\infty}(ab)_n &= \Big(\lim_{n\to\infty}a_n\Big)\Big(\lim_{n\to\infty}b_n\Big),\\
\lim_{n\to\infty}(\max((a,b)))_n &= \max\Big(\lim_{n\to\infty}a_n, \lim_{n\to\infty}b_n\Big),\\
\lim_{n\to\infty}(-a)_n &= -\lim_{n\to\infty}a_n, \text{ and}\\
\lim_{n\to\infty}|a|_n &= \Big|\lim_{n\to\infty}a_n\Big|.
\end{aligned}$$

Further, if $\lim_{n\to\infty}a_n \neq 0_{\mathbb{R}}$ and for each $i \geq m$, we have $a_i \neq 0_{\mathbb{R}}$, then $((a^{-1})_i)_{i=m}^{\infty}$ is a convergent sequence of reals with

$$\lim_{n\to\infty}(a^{-1})_n = (\lim_{n\to\infty}a_n)^{-1}.$$

Corollary 6.1.47. *Let $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ be convergent sequences of reals. Then $((a-b)_i)_{i=m}^{\infty}$, $((\min((a,b)))_i)_{i=m}^{\infty}$ are convergent sequences of reals with*

$$\begin{aligned}
\lim_{n\to\infty}(a-b)_n &= \lim_{n\to\infty}a_n - \lim_{n\to\infty}b_n, \text{ and}\\
\lim_{n\to\infty}(\min((a,b)))_n &= \min\Big(\lim_{n\to\infty}a_n, \lim_{n\to\infty}b_n\Big).
\end{aligned}$$

Further, if $\lim_{n\to\infty}b_n \neq 0_{\mathbb{R}}$ and for each $i \geq m$, we have $b_i \neq 0_{\mathbb{R}}$, then $((a/b)_i)_{i=m}^{\infty}$ is a convergent sequence of reals with

$$\lim_{n\to\infty}\left(\frac{a}{b}\right)_n = \frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n}.$$

Lemma 6.1.48 (Raising real sequences to powers)**.** *Let a be a function, $n \in \mathbb{N}$, $p \in \mathbb{Z}$ and $r \in \mathbb{Q}$ such that there exists an $m \in \mathbb{Z}$ so that $(a_i)_{i=m}^{\infty}$ is a sequence of reals. Set $K \coloneqq \operatorname{SeqInt}_{\mathbb{R}} a$. Then*

(i) there exists a unique function b such that $(b_i)_{i=K}^{\infty}$ is a sequence of reals so that $b_i = a_i^n$ for each $i \geq K$,

(ii) $a_i \neq 0_{\mathbb{R}}$ *for any* $i \geq K \implies$ *there exists a unique function* c *such that* $(c_i)_{i=K}^{\infty}$ *is a sequence of reals so that* $c_i = a_i^p$ *for each* $i \geq K$*,*

(iii) $a_i \geq 0_{\mathbb{R}}$ *for each* $i \geq K$ *and* $n \geq 1 \implies$ *there exists a unique function* d *such that* $(d_i)_{i=K}^{\infty}$ *is a sequence of reals so that* $d_i = a_i^{1/n}$ *for each* $i \geq K$*, and*

(iv) $a_i > 0_{\mathbb{R}}$ *for each* $i \geq K \implies$ *there exists a unique function* e *such that* $(e_i)_{i=K}^{\infty}$ *is a sequence of reals and* $e_i = a_i^r$ *for each* $i \geq K$*.*

Remark 6.1.49. This allows to denote b, c, d, e by a^n, a^p, $a^{1/n}$, a^r respectively. Axiom of substitution obeyed.

Corollary 6.1.50 (Continuity for natural and integer exponentiation)**.** *Let* $(a_i)_{i=m}^{\infty}$ *be a convergent sequence of reals, and* $n \in \mathbb{N}$ *and* $p \in \mathbb{Z}$*. Then*

(i) $((a^n)_i)_{i=m}^{\infty}$ *is a convergent sequence of reals with*

$$\lim_{i\to\infty} (a^n)_i = \left(\lim_{i\to\infty} a_i\right)^n, \text{ and}$$

(ii) $a_i \neq 0_{\mathbb{R}}$ *for any* $i \geq m$*, and* $\lim_{i\to\infty} a_i \neq 0_{\mathbb{R}} \implies ((a^p)_i)_{i=m}^{\infty}$ *is a convergent sequence of reals with*

$$\lim_{i\to\infty} (a^p)_i = \left(\lim_{i\to\infty} a_i\right)^p.$$

Lemma 6.1.51. *Let* $n \geq 1$ *and* $(a_i)_{i=m}^{\infty}$ *converge to* $1_{\mathbb{R}}$ *in* $\mathbb{R}$ *and* $a_i \geq 0_{\mathbb{R}}$ *for each* $i \geq m$*. Then* $((a^{1/n})_i)_{i=m}^{\infty}$ *also converges to* $1_{\mathbb{R}}$ *in* $\mathbb{R}$*.*

Corollary 6.1.52 (Continuity of exponentiation by n-th roots and rationals)**.** *Let* $(a_i)_{i=m}^{\infty}$ *be a convergent sequence of reals, and* $n \geq 1$ *and* $r \in \mathbb{Q}$*. Then*

(i) $a_i \geq 0_{\mathbb{R}}$ *for each* $i \geq m \implies \lim_{n\to\infty} a_n \geq 0_{\mathbb{R}}$ *and* $((a^{1/n})_i)_{i=m}^{\infty}$ *is a convergent sequence of reals with*

$$\lim_{i\to\infty} (a^{1/n})_i = \left(\lim_{i\to\infty} a_i\right)^{1/n}, \text{ and}$$

(ii) $a_i > 0_{\mathbb{R}}$ *for each* $i \geq m$*, and* $\lim_{n\to\infty} a_n > 0_{\mathbb{R}} \implies ((a)^r{}_i)_{i=m}^{\infty}$ *is a convergent sequence of reals with*

$$\lim_{i\to\infty} (a^r)_i = \left(\lim_{i\to\infty} a_i\right)^r.$$

Lemma 6.1.53. *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals such that $a_i \neq 0_{\mathbb{R}}$ for any $i \geq m$. Let $(a_i)_{i=m}^{\infty}$ converge to $0_{\mathbb{R}}$ in $\mathbb{R}$. Then $((a^{-1})_i)_{i=m}^{\infty}$ is a divergent sequence of reals.*

Definition 6.1.54 (Strictly increasing sequences of reals)**.** For any objects a, m, we write "$(a_i)_{i=m}^{\infty}$ is a strictly increasing sequence of reals" iff for $(a_i)_{i=m}^{\infty}$ is a sequence of reals and for each $i \geq m$, we have $i + 1_{\mathbb{Z}} \geq m$ and $a_{i+1_{\mathbb{Z}}} > a_i$.

Remark 6.1.55. Axiom of substitution obeyed.

Lemma 6.1.56 (Characterization of strictly increasing sequences of reals)**.** *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals. Then $(a_i)_{i=m}^{\infty}$ is strictly increasing $\iff$ for all $i, j \geq m$, we have that $i < j \implies a_i < a_j$.*

6.2 The extended real number system

July 19, 2021

Axiom 6.1 (Infinities)**.**

(i) Infinities are objects.
(ii) $+\infty$, $-\infty$ are infinities.
(iii) $+\infty, -\infty \notin \mathbb{R}$.
(iv) $+\infty \neq -\infty$.

Remark 6.2.1. We set $\mathbb{R}^* \coloneqq \mathbb{R} \cup \{+\infty, -\infty\}$.

Lemma 6.2.2 (Negation on $\mathbb{R}^*$)**.** *Let $x \in \mathbb{R}^*$. Then there exists a unique $y \in \mathbb{R}^*$ such that one the following holds:*

(i) $x = -\infty$ and $y = +\infty$.
(ii) $x \in \mathbb{R}$ and $y = -x$.
(iii) $x = +\infty$ and $y = -\infty$.

Lemma 6.2.3. *Let $x \in \mathbb{R}$. Then $x \in \mathbb{R}^*$, and for any $y \in \mathbb{R}^*$ such that one of the above holds, $y = -x$.*

Remark 6.2.4. Lemmas 6.2.2 and 6.2.3 allow to denote y of Lemma 6.2.2 by $-x$. Axiom of substitution obeyed.

Corollary 6.2.5. $-(+\infty) = -\infty$ *and* $-(-\infty) = +\infty$.

Lemma 6.2.6 (Double negation)**.** *Let $x \in \mathbb{R}$. Then $-x \in \mathbb{R}^*$ and $-(-x) = x$.*

Definition 6.2.7 (Order on $\mathbb{R}^*$)**.**
- (i) For any objects x, y, we write "$x < y$", or "$y > x$", iff one of the following holds:
  - (a) $x = -\infty$ and $y \in \mathbb{R}^* \setminus \{-\infty\}$.
  - (b) $x \in \mathbb{R}$ and $x < y$.
  - (c) $x \in \mathbb{R}^* \setminus \{+\infty\}$ and $y = +\infty$.
- (ii) For any objects x, y, we write "$x \leq y$", or "$y \geq x$", iff $x < y$ or ($x \in \mathbb{R}$ and $x = y$) or (x is an infinity and $x = y$).

Remark 6.2.8. Axiom of substitution obeyed. Again, a different symbol should've been used.

Lemma 6.2.9 (Characterizing " $\leq$ " on $\mathbb{R}^*$)**.** *Let $x, y \in \mathbb{R}^*$. Then $x \leq y \iff$ one of the following holds:*
- *(i) $x = -\infty$.*
- *(ii) $x, y \in \mathbb{R}$ and $x \leq y$.*
- *(iii) $y = +\infty$.*

Proposition 6.2.10 (Properties of order on $\mathbb{R}^*$)**.** *Let $x, y, z \in \mathbb{R}^*$. Then*
- *(i)* (transitivity) *$x < y$ and $y < z \implies x < z$,*
- *(ii)* (negation reverses order) *$x < y \implies -x > -y$, and*
- *(iii)* (trichotomy) *exactly one of these holds: $x < y$, $x = y$, or $x > y$.*

Lemma 6.2.11 (Suprema of subsets of $\mathbb{R}^*$)**.** *Let $E \subseteq \mathbb{R}^*$. Then there exists a unique $y \in \mathbb{R}^*$ such that one of the following holds:*
- *(i) $E = \emptyset$ and $y = -\infty$.*
- *(ii) $E \neq \emptyset$ and*
  - *(a) $+\infty \in E$ and $y = +\infty$.*
  - *(b) $+\infty \notin E$ and*
    - *(I) $-\infty \in E$ and*
      - *(A) $E \setminus \{-\infty\} = \emptyset$ and $y = -\infty$.*
      - *(B) $E \setminus \{-\infty\} \neq \emptyset$ and*
        - *(1) there exists an $M \in \mathbb{R}$ such that M is an upper bound for $E \setminus \{-\infty\}$, and y is the least upper bound for $E \setminus \{-\infty\}$.*

(2) there exists no $M \in \mathbb{R}$ such that M is an upper bound for $E \setminus \{-\infty\}$, and $y = +\infty$.

(II) $-\infty \notin E$ and

(A) there exists an $M \in \mathbb{R}$ such that M is an upper bound for E, and y is the least upper bound for E.

(B) there exists no $M \in \mathbb{R}$ such that M is an upper bound for E, and $y = +\infty$.

Further, for any $y \in \mathbb{R}^*$, exactly one of the above holds.

Remark 6.2.12. This allows to denote y by $\sup(E)$. Axiom of substitution obeyed.

Lemma 6.2.13 (Infima of subsets of $\mathbb{R}^*$)**.** *Let $E \subseteq \mathbb{R}^*$. Then there exists a unique $y \in \mathbb{R}^*$ such that one of the following holds:*

(i) $E = \emptyset$ and $y = +\infty$.

(ii) $E \neq \emptyset$ and

(a) $-\infty \in E$ and $y = -\infty$.

(b) $-\infty \notin E$ and

(I) $+\infty \in E$ and

(A) $E \setminus \{+\infty\} = \emptyset$ and $y = +\infty$.

(B) $E \setminus \{+\infty\} \neq \emptyset$ and

(1) there exists an $M \in \mathbb{R}$ such that M is a lower bound for $E \setminus \{+\infty\}$, and y is the greatest lower bound for $E \setminus \{+\infty\}$.

(2) there exists no $M \in \mathbb{R}$ such that M is a lower bound for $E \setminus \{+\infty\}$, and $y = -\infty$.

(II) $+\infty \notin E$ and

(A) there exists an $M \in \mathbb{R}$ such that M is a lower bound for E, and y is the greatest lower bound for E.

(B) there exists no $M \in \mathbb{R}$ such that M is an lower bound for E, and $y = -\infty$.

Further, for any $y \in \mathbb{R}^*$, exactly one of the above holds.

Remark 6.2.14. This allows to denote y by $\inf(E)$. Axiom of substitution obeyed.

Remark 6.2.15. Remark 5.5.10 allows to set $-E := \{-x : x \in E\}$ for any subset $E \subseteq \mathbb{R}^*$.

Lemma 6.2.16. *Let* $E \subseteq \mathbb{R}^*$. *Then* $-E \subseteq \mathbb{R}^*$ *and* $-(-E) = E$.

Lemma 6.2.17 (Relating suprema and infima)**.** *Let* $E \subseteq \mathbb{R}^*$. *Then*

(i) $\sup(E) = -\inf(-E)$, *and*
(ii) $\inf(E) = -\sup(-E)$.

Proposition 6.2.18. *(i) Set* $E := \{-i_{\mathbb{R}} : i \geq 1\} \cup \{-\infty\}$. *Then* $\sup(E) = -1_{\mathbb{R}}$ *and* $\inf(E) = -\infty$.

(ii) Let $r > 1_{\mathbb{R}}$ *and set* $F := \{r^{-n} : n \geq 1_{\mathbb{Z}}\}$. *Then* $\sup(F) = r^{-1}$ *and* $\inf(F) = 0_{\mathbb{R}}$.

(iii) Set $G := \{n_{\mathbb{R}} : n \geq 1\}$. *Then* $\sup(E) = +\infty$ *and* $\inf(E) = 1_{\mathbb{R}}$.

(iv) $\sup(\emptyset) = -\infty$ *and* $\inf(\emptyset) = +\infty$.

Theorem 6.2.19 (Properties of suprema and infima)**.** *Let* $E \subseteq \mathbb{R}^*$ *and* $M \in \mathbb{R}^*$. *Then*

(i) for each $x \in E$, *we have* $\inf(E) \leq x \leq \sup(E)$,
(ii) $x \leq M$ *for each* $x \in E \implies M \geq \sup(E)$, *and*
(iii) $x \geq M$ *for each* $x \in E \implies M \leq \inf(E)$.

Corollary 6.2.20. *Let* $E \subseteq \mathbb{R}^*$ *such that* $\sup(E) < \inf(E)$. *Then* $E = \emptyset$.

Lemma 6.2.21. *Let* $E \subset \mathbb{R}^*$. *Then*

(i) $\sup(E) = -\infty \iff E = \emptyset$ *or* $E = \{-\infty\}$, *and*
(ii) $\inf(E) = +\infty \iff E = \emptyset$ *or* $E = \{+\infty\}$.

Lemma 6.2.22 (Nonempty bounded subsets of $\mathbb{R}$ have real sup's and inf's)**.** *Let* $E \subseteq \mathbb{R}$ *such that* $E \neq \emptyset$ *and there exists an* $M \geq 0_{\mathbb{R}}$ *such that* $|x| \leq M$ *for each* $x \in E$. *Then* $\sup(E), \inf(E) \in \mathbb{R}$.

Lemma 6.2.23 (sup's and inf's of subsets)**.** *Let* $A, B \subseteq \mathbb{R}^*$ *such that* $A \subseteq B$. *Then* $\sup(A) \leq \sup(B)$ *and* $\inf(A) \geq \inf(B)$.

6.3 Suprema and infima of sequences

July 26, 2021

Remark 6.3.1. Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals. Then we set $\sup\,(a_i)_{i=m}^{\infty} := \sup(\{a_i : i \geq m\})$ and $\inf\,(a_i)_{i=m}^{\infty} := \inf(\{a_i : i \geq m\})$.

Proposition 6.3.2. *(i) There exists a unique function a such that* $(a_i)_{i=1_\mathbb{Z}}^\infty$ *is a sequence of reals so that* $a_i = (-1_\mathbb{R})^i$ *for each* $i \geq 1_\mathbb{Z}$*. Further, for any such a, we have* $\sup(a_i)_{i=1_\mathbb{Z}}^\infty = 1_\mathbb{R}$ *and* $\inf(a_i)_{i=1_\mathbb{Z}}^\infty = -1_\mathbb{R}$*.*

(ii) There exists a unique function b such that $(b_i)_{i=1_\mathbb{Z}}^\infty$ *is a sequence of reals so that* $b_i = (i_{\mathbb{R}'})^{-1}$ *for each* $i \geq 1_\mathbb{Z}$*. Further for any such b, we have* $\sup(b_i)_{i=1_\mathbb{Z}}^\infty = 1_\mathbb{R}$ *and* $\inf(b_i)_{i=1_\mathbb{Z}}^\infty = 0_\mathbb{R}$*.*

(iii) There exists a unique function c such that $(c_i)_{i=1_\mathbb{Z}}^\infty$ *is a sequence of reals so that* $c_i = i_{\mathbb{R}'}$ *for each* $i \geq 1_\mathbb{Z}$*. Further for any such c, we have* $\sup(c_i)_{i=1_\mathbb{Z}}^\infty = +\infty$ *and* $\inf(c_i)_{i=1_\mathbb{Z}}^\infty = 1_\mathbb{R}$*.*

Lemma 6.3.3 (Bounded sequences of reals have real sup's and inf's)**.** *Let* $(a_i)_{i=m}^\infty$ *be a bounded sequence of reals. Then* $\sup(a_i)_{i=m}^\infty, \inf(a_i)_{i=m}^\infty \in \mathbb{R}$*.*

Proposition 6.3.4 (Properties of sup and inf of sequences of extended reals)**.** *Let* $(a_i)_{i=m}^\infty$ *be a sequence of reals and* $M \in \mathbb{R}^*$*. Set* $x := \sup(a_i)_{i=m}^\infty$ *and* $y := \inf(a_i)_{i=m}^\infty$*. Then*

(i) $x \geq a_i \geq y$ *for all* $i \geq m$*,*
(ii) $a_i \leq M$ *for all* $i \geq m \implies M \geq x$*,*
(iii) $a_i \geq M$ *for all* $i \geq m \implies M \leq y$*,*
(iv) $M < x \implies a_i > M$ *for some* $i \geq m$*, and*
(v) $M > y \implies a_i < M$ *for some* $i \geq M$*.*

Definition 6.3.5 (Increasing, decreasing and monotone sequences of reals)**.** For any objects a, m, we write

(i) "$(a_i)_{i=m}^\infty$ is an increasing sequence of reals" iff $(a_i)_{i=m}^\infty$ is a sequence of reals and for each $i \geq m$, we have $i + 1_\mathbb{Z} \geq m$ and $a_{i+1_\mathbb{Z}} \geq a_i$,
(ii) "$(a_i)_{i=m}^\infty$ is a decreasing sequence of reals" iff $(a_i)_{i=m}^\infty$ is a sequence of reals and for each $i \geq m$, we have $i + 1_\mathbb{Z} \geq m$ and $a_{i+1_\mathbb{Z}} \leq a_i$, and
(iii) "$(a_i)_{i=m}^\infty$ is a monotone sequence of reals" iff $(a_i)_{i=m}^\infty$ is an increasing sequence of reals, or $(a_i)_{i=m}^\infty$ is s decreasing sequence of reals.

Remark 6.3.6. Axiom of substitution obeyed by all.

Lemma 6.3.7 (Characterization of increasing and decreasing sequences)**.** *Let* $(a_i)_{i=m}^\infty$ *be a sequence of reals. Then*

(i) the following are equivalent:

  (a) $(a_i)_{i=m}^\infty$ *is an increasing sequence of reals;*
  (b) for each $i, j \geq m$*, we have that* $i < j \implies a_i \leq a_j$*;*
  (c) $((-a)_i)_{i=m}^\infty$ *is a decreasing sequence of reals; and*

(ii) the following are equivalent:

(a) $(a_i)_{i=m}^{\infty}$ *is a decreasing sequence of reals;*

(b) for each $i, j \geq m$*, we have that* $i < j \implies a_i \geq a_j$*;*

(c) $((-a)_i)_{i=m}^{\infty}$ *is an increasing sequence of reals.*

Proposition 6.3.8 (Monotone bounded sequences converge)**.** *(i) Let* $(a_i)_{i=m}^{\infty}$ *be an increasing sequence of reals and* $M \in \mathbb{R}$ *such that* $a_i \leq M$ *for each* $i \geq m$*. Then* $(a_i)_{i=m}^{\infty}$ *is a convergent sequence of reals, and* $\sup (a_i)_{i=m}^{\infty} \in \mathbb{R}$ *and* $\lim_{n\to\infty} a_n = \sup (a_i)_{i=m}^{\infty} \leq M$*.*

(ii) Let $(a_i)_{i=m}^{\infty}$ *be a decreasing sequence of reals and* $M \in \mathbb{R}$ *such that* $a_i \geq M$ *for each* $i \geq m$*. then* $(a_i)_{i=m}^{\infty}$ *is a convergent sequence of reals, and* $\inf (a_i)_{i=m}^{\infty} \in \mathbb{R}$ *and* $\lim_{n\to\infty} a_n = \inf (a_i)_{i=m}^{\infty} \geq M$*.*

Corollary 6.3.9. *Let* $(a_i)_{i=m}^{\infty}$ *be a monotone sequence of reals. Then* $(a_i)_{i=m}^{\infty}$ *is a convergent sequence of reals* $\iff$ $(a_i)_{i=m}^{\infty}$ *is a bounded sequence of reals.*

Lemma 6.3.10 (Constant sequences of reals converge)**.** *Let* $x \in \mathbb{R}$ *and* $m \in \mathbb{Z}$*. Then there exists a unique function a such that* $(a_i)_{i=m}^{\infty}$ *is a sequence of reals such that* $a_i = x$ *for each* $i \geq M$*. Further, for any such a, we have that* $(a_i)_{i=m}^{\infty}$ *converges to x in* $\mathbb{R}$*.*

Remark 6.3.11. This allows to denote a by $\mathrm{Seq}_{\mathbb{R},\mathrm{m}} x$. Axiom of substitution obeyed.

Lemma 6.3.12 (Consistency with embedding for constant sequences)**.** *Let* $r \in \mathbb{Q}$*. Then* $\mathrm{Seq}_{\mathbb{Q}}\, r_{\mathbb{R}''} = \mathrm{Seq}_{\mathbb{R},1_{\mathbb{Z}}} r_{\mathbb{R}''}$*.*

Proposition 6.3.13. *(i)* $(0 < x < 1 \implies \lim_{n\to\infty} x^n = 0)$**.** *Let* $0_{\mathbb{R}} < x < 1_{\mathbb{R}}$*. Then there exists a unique function a such that* $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *is a sequence of reals so that* $a_i = x^i$ *for each* $i \geq 1_{\mathbb{Z}}$*. Further, for any such a, we have that* $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *is a convergent sequence of reals with* $\lim_{n\to\infty} a_n = 0_{\mathbb{R}}$*.*

(ii) $(x > 1 \implies (x^n)_{n=1}^{\infty}$ diverges$)$**.** *Let* $x > 1_{\mathbb{R}}$*. Then there exists a unique function a such that* $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *is a sequence of reals so that* $a_i = x^i$ *for each* $i \geq 1_{\mathbb{Z}}$*. Further, for any such a, we have that* $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ *is a divergent sequence of reals.*

6.4 Limsup, liminf and limit points

July 31, 2021

Definition 6.4.1 (Limit points of real sequences)**.** For any objects x, a, m, we write "x is a limit point of $(a_i)_{i=m}^{\infty}$" iff $(a_i)_{i=m}^{\infty}$ is a sequence of reals, and one of the following holds:

(i) $x = -\infty$ and $\inf (a_i)_{i=m}^{\infty} = -\infty$.
(ii) $x \in \mathbb{R}$ and for every $\varepsilon > 0_{\mathbb{R}}$ and for every $N \geq m$, there exists an $i \geq N$ such that $|a_i - x| \leq \varepsilon$.
(iii) $x = +\infty$ and $\sup (a_i)_{i=m}^{\infty} = +\infty$.

Remark 6.4.2. Axiom of substitution obeyed.

Proposition 6.4.3 (Limit of a convergent sequence is its only limit point)**.** *Let $(a_i)_{i=m}^{\infty}$ be a convergent sequence of reals and $L \in \mathbb{R}$. Then L is a limit point of $(a_i)_{i=m}^{\infty} \iff L = \lim_{n\to\infty} a_n$.*

Remark 6.4.4. Let a be a function such that there exists an $m \in \mathbb{Z}$ such that $(a_i)_{i=m}^{\infty}$ be a sequence of reals. Then for each $N \geq \mathrm{SeqInt}_{\mathbb{R}}\, a$, we set

$$a_N^+ \coloneqq \sup(\{a_i : i \geq N\}), \text{ and}$$
$$a_N^- \coloneqq \inf(\{a_i : i \geq N\}).$$

Further, we set

$$\limsup_{n\to\infty} a_n \coloneqq \inf(\{a_N^+ : N \geq \mathrm{SeqInt}_{\mathbb{R}}\, a\}), \text{ and}$$
$$\liminf_{n\to\infty} a_n \coloneqq \sup(\{a_N^- : N \geq \mathrm{SeqInt}_{\mathbb{R}}\, a\}).$$

Lemma 6.4.5. *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals and $M, N \geq m$. Then $a_M^+ \leq a_N^+$ and $a_M^- \geq a_N^-$.*

Proposition 6.4.6 (Limit points, lim sup's and lim inf's are insensitive to finite initial conditions)**.** *Let $(a_i)_{i=m}^{\infty}$, $(a_i)_{i=m}^{\infty}$ be sequences of reals, and $x \in \mathbb{R}^*$ and let one of the following holds:*

(i) There exists an $N \geq m, n$ such that $a_i = b_i$ for each $i \geq N$.
(ii) There exists a $k \geq 0_{\mathbb{Z}}$ such that for each $i \geq m$, we have $i + k \geq n$ and $a_i = b_{i+k}$.

Then

(i) x is a limit point of $(a_i)_{i=m}^{\infty} \iff x$ is a limit point of $(b_i)_{i=n}^{\infty}$,
(ii) $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n$, and
(iii) $\liminf_{n\to\infty} a_n = \liminf_{n\to\infty} b_n$.

Lemma 6.4.7. *Let $(a_i)_{i=m}^{\infty}$ be a bounded sequence of reals. Then* $\limsup_{n\to\infty} a_n, \liminf_{n\to\infty} a_n \in \mathbb{R}$.

Proposition 6.4.8 (Properties of lim sup's and lim inf's)**.** *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals, and $x \in \mathbb{R}^*$ and set $L^+ := \sup (a_i)_{i=m}^{\infty}$ and $L^- := \inf (a_i)_{i=m}^{\infty}$. Then*

(i) (a) $x > L^+ \implies$ *there exists an $N \geq M$ such that $a_i < x$ for each $i \geq N$,*
  (b) $x < L^- \implies$ *there exists an $N \geq m$ such that $a_i > x$ for each $i \geq N$,*

(ii) (a) $x < L^+ \implies$ *for each $N \geq m$, there exists an $i \geq N$ such that $a_i > x$,*
  (b) $x > L^- \implies$ *for each $N \geq m$, there exists an $i \geq N$ such that $a_i < x$,*

(iii) $\inf (a_i)_{i=m}^{\infty} \leq L^- \leq L^+ \leq \sup (a_i)_{i=m}^{\infty}$,

(iv) *x is a limit point of $(a_i)_{i=m}^{\infty}$* $\implies L^- \leq x \leq L^+$,

(v) *L^+, L^- are limit points of $(a_i)_{i=m}^{\infty}$, and*

(vi) $x \in \mathbb{R} \implies$ *the following are equivalent:*
  (a) *$(a_i)_{i=m}^{\infty}$ converges to x in $\mathbb{R}$.*
  (b) $L^- = x = L^+$.

Proposition 6.4.9 (Limit points of real limit points are limit points)**.** *Let $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ be sequences of reals such that b_i is a limit point of $(a_i)_{i=m}^{\infty}$ for each $i \geq m$. Let c be a limit point of $(b_i)_{i=m}^{\infty}$. Then c is also a limit point of $(a_i)_{i=m}^{\infty}$.*

Lemma 6.4.10 (Comparison principles)**.** *Let $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$ be sequences of reals such that $a_i \leq b_i$ for each $i \geq m$. Then*

(i) *for each $N \geq m$, we have $a_N^+ \leq b_N^+$ and $a_N^- \leq b_N^-$,*

(ii) $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$, *and*

(iii) $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Corollary 6.4.11 (Squeeze test)**.** *Let $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=m}^{\infty}$, $(c_i)_{i=m}^{\infty}$ be sequences of reals such that $(a_i)_{i=m}^{\infty}$, $(c_i)_{i=m}^{\infty}$ are convergent sequences of reals, and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ *and,* $\min(a_i, c_i) \leq b_i \leq \max(a_i, c_i)$ *for each $i \geq m$. Then $(b_i)_{i=m}^{\infty}$ is a convergent sequence of reals with* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$.

Corollary 6.4.12 (Zero test)**.** *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals. Then $(a_i)_{i=m}^{\infty}$ converges to $0_{\mathbb{R}}$ in $\mathbb{R}$ $\iff$ $(|a|_i)_{i=m}^{\infty}$ converges to $0_{\mathbb{R}}$.*

Theorem 6.4.13 (Completeness of $\mathbb{R}$)**.** *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals. Then $(a_i)_{i=m}^{\infty}$ is a convergent sequence of reals $\iff$ $(a_i)_{i=m}^{\infty}$ is a Cauchy sequence of reals.*

6.5 Some standard limits

August 4, 2021

Proposition 6.5.1 ($\lim_{n\to\infty} 1/n^{1/k} = 0$ for natural $k \geq 1$)**.** *Let $k \geq 1$. Then there exists a unique function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a sequence of reals so that $a_i = ((i_{\mathbb{R}'})^{-1})^{1/k}$ for each $i \geq 1_{\mathbb{Z}}$. Further, for any such function a, we have that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ converges to $0_{\mathbb{R}}$ in $\mathbb{R}$.*

Proposition 6.5.2 (Convergence of $(x^n)_{n=1}^{\infty}$ for $x \in \mathbb{R}$)**.** *Let $x \in \mathbb{R}$. Then there exists a unique function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a sequence of reals so that for each $i \geq 1_{\mathbb{Z}}$, there exists an $j \geq 1$ such that $i = j_{\mathbb{Z}}$ and $a_i = x^j$. Further, for any such function a, we have that*

(i) $|x| < 1_{\mathbb{R}} \implies (a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ converges to $0_{\mathbb{R}}$ in $\mathbb{R}$,
(ii) $|x| > 1_{\mathbb{R}} \implies (a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a divergent sequence of reals,
(iii) $x = 1_{\mathbb{R}} \implies (a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ converges to $1_{\mathbb{R}}$ in $\mathbb{R}$, and
(iv) $x = -1_{\mathbb{R}} \implies (a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a divergent sequence of reals.

Proposition 6.5.3 ($\lim_{n\to\infty} x^{1/n} = 1$ for real $x \geq 0$)**.** *Let $x > 0_{\mathbb{R}}$. Then there exists a unique function a such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a sequence of reals so that $a_i = x^{1_{\mathbb{Q}}/i_{\mathbb{Q}'}}$ for each $i \geq 1_{\mathbb{Z}}$. Further, for any such function a, we have that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ converges to $1_{\mathbb{R}}$ in $\mathbb{R}$.*

Remark 6.5.4. The case when $x = 0_{\mathbb{R}}$ is simple.

Proposition 6.5.5 ($\lim_{n\to\infty} 1/n^q = 0$ while $(n^q)_{n=1}^{\infty}$ diverges for rational $q > 0$)**.** *Let $q > 0_{\mathbb{Q}}$. Then there exist unique functions a, b such that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$, $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$ are sequences of reals so that for each $i \geq 1_{\mathbb{Z}}$, we have $a_i = ((i_{\mathbb{R}'})^{-1})^q$ and $b_i = (i_{\mathbb{R}'})^q$. Further, for any such functions a, b, we have that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ converges to $0_{\mathbb{R}}$ in $\mathbb{R}$, while $(b_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a divergent sequence of reals.*

6.6 Subsequences

August 7, 2021

Definition 6.6.1 (Strictly increasing functions on initial segments of $\mathbb{Z}$)**.** For any bjects f, m, n, we write "f is a strictly increasing function from $\{i \in \mathbb{Z} : i \geq m\}$ to $\{i \in \mathbb{Z} : i \geq n\}$" iff $m, n \in \mathbb{Z}$, and $f\colon \{i \in \mathbb{Z} : i \geq m\} \to \{i \in \mathbb{Z} : i \geq n\}$ and for each $i \geq m$, we have $i + 1_{\mathbb{Z}} \geq m$ and $f(i + 1_{\mathbb{Z}}) > f(i)$.

Remark 6.6.2. Axiom of substitution obeyed.

Lemma 6.6.3 (Characterizing strictly increasing functions on initial segments on $\mathbb{Z}$)**.** *Let $m, n \in \mathbb{Z}$ and $f\colon \{i \in \mathbb{Z} : i \geq m\} \to \{i \in \mathbb{Z} : i \geq n\}$. Then f is an increasing function from $\{i \in \mathbb{Z} : i \geq m\}$ to $\{i \in \mathbb{Z} : i \geq n\}$ $\iff$ for each $i, j \geq m$, we have that $i > j \implies f(i) > f(j)$.*

Lemma 6.6.4. *Let f be a strictly increasing function from $\{i \in \mathbb{Z} : i \geq m\}$ to $\{i \in \mathbb{Z} : i \geq n\}$ and let $N \in \mathbb{N}$. Then $f(m + N_{\mathbb{Z}}) \geq n + N_{\mathbb{Z}}$.*

Definition 6.6.5 (Subsequences of real sequences)**.** For any objects a, b, m, n, we write "$(a_i)_{i=m}^{\infty}$ is a subsequence of $(b_i)_{i=n}^{\infty}$ in $\mathbb{R}$" iff $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=n}^{\infty}$ are sequences of reals and there exists an object f such that f is a strictly increasing function from $\{i \in \mathbb{Z} : i \geq m\}$ to $\{i \in \mathbb{Z} : i \geq n\}$ such that for each $i \geq m$, we have $f(i) \geq n$ and $a_i = b_{f(i)}$.

Remark 6.6.6. Axiom of substitution obeyed.

Proposition 6.6.7. *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals such that for each $M \in \mathbb{R}$, there exists an $i \geq m$ such that $|a_i| > M$. Then there exist b, n such that $(b_i)_{i=n}^{\infty}$ is a subsequence of $(a_i)_{i=m}^{\infty}$ in $\mathbb{R}$ so that $b_i \neq 0_{\mathbb{R}}$ for each $i \geq n$, and $((b^{-1})_i)_{i=n}^{\infty}$ converges to $0_{\mathbb{R}}$ in $\mathbb{R}$.*

Lemma 6.6.8 ("is a subsequence of" is reflexive and transitive)**.** *Let $(a_i)_{i=m}^{\infty}$, $(b_i)_{i=n}^{\infty}$, $(c_i)_{i=k}^{\infty}$ be sequences of reals. Then*

(i) (reflexivity) *$(a_i)_{i=m}^{\infty}$ is a subsequence of $(a_i)_{i=m}^{\infty}$ in $\mathbb{R}$, and*
(ii) (transitivity) *$(a_i)_{i=m}^{\infty}$ is a subsequence of $(b_i)_{i=n}^{\infty}$ in $\mathbb{R}$ and $(b_i)_{i=n}^{\infty}$ is a subsequence of $(c_i)_{i=k}^{\infty}$ in $\mathbb{R}$ $\implies$ $(a_i)_{i=m}^{\infty}$ is a subsequence of $(c_i)_{i=k}^{\infty}$ in $\mathbb{R}$.*

Proposition 6.6.9 (Subsequences and limits)**.** *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals and $L \in \mathbb{R}$. Then the following are equivalent:*

(i) $(a_i)_{i=m}^{\infty}$ converges to L in $\mathbb{R}$.
(ii) For any b, n, we have that $(b_i)_{i=n}^{\infty}$ is a subsequence of $(a_i)_{i=m}^{\infty}$ in $\mathbb{R}$ $\implies$ $(b_i)_{i=n}^{\infty}$ converges to L in $\mathbb{R}$.

Proposition 6.6.10 (Subsequences and limit points). *Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals and $L \in \mathbb{R}$. Then the following are equivalent:*

(i) *L is a limit point of $(a_i)_{i=m}^{\infty}$.*
(ii) *There exist b, n such that $(b_i)_{i=n}^{\infty}$ is a subsequence of $(a_i)_{i=m}^{\infty}$ in $\mathbb{R}$ and $(b_i)_{i=n}^{\infty}$ converges to L in $\mathbb{R}$.*

Theorem 6.6.11 (Bolzano-Weierstrass). *Let $(a_i)_{i=m}^{\infty}$ be a bounded sequence of reals. Then there exist b, n such that $(b_i)_{i=n}^{\infty}$ is a subsequence of $(a_i)_{i=m}^{\infty}$ in $\mathbb{R}$ and $(b_i)_{i=n}^{\infty}$ is a convergent sequence of reals.*

6.7 Real exponentiation, part II

August 17, 2021

Lemma 6.7.1 (Raising reals to rational sequences). *Let $x > 0_{\mathbb{R}}$ and a be a function such that there exists an m so that $(a_i)_{i=m}^{\infty}$ be a sequence of rationals. Then there exists a unique function b such that $(b_i)_{i=\mathrm{SeqInt}_{\mathbb{Q}}\, a}^{\infty}$ is a sequence of reals and $b_i = x^{a_i}$ for each $i \geq \mathrm{SeqInt}_{\mathbb{Q}}\, a$.*

Remark 6.7.2. This allows to denote b by x^a. Axiom of substitution obeyed.

Lemma 6.7.3. *Let $x > 0_{\mathbb{R}}$ and $(a_i)_{i=m}^{\infty}$ be a Cauchy sequence of rationals. Then $((x^a)_i)_{i=m}^{\infty}$ is a convergent sequence of reals with $\lim_{n\to\infty} (x^a)_n > 0_{\mathbb{R}}$.*

Lemma 6.7.4. *Let $x > 0$ and $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ be a Cauchy sequence of rationals such that $\mathrm{LIM}_{n\to\infty} a_n = 0_{\mathbb{R}}$. Then $\lim_{n\to\infty} (x^c)_n$ converges to $1_{\mathbb{R}}$ in $\mathbb{R}$.*

Corollary 6.7.5 (Raising positive reals to real exponents). *Let $x > 0_{\mathbb{R}}$ and $\alpha \in \mathbb{R}$. Then there exists a unique $y \in \mathbb{R}$ such that there exists a function a so that $(a_i)_{i=1_{\mathbb{Z}}}^{\infty}$ is a Cauchy sequence of rationals such that $\mathrm{LIM}_{n\to\infty} a_n = \alpha$, and $((x^a)_i)_{i=1_{\mathbb{Z}}}^{\infty}$ converges to y in $\mathbb{R}$.*

Remark 6.7.6. This allows to denote y by x^α. Axiom of substitution obeyed.

Corollary 6.7.7. *Let $x > 0_{\mathbb{R}}$ and $\alpha \in \mathbb{R}$. Then $x^\alpha > 0_{\mathbb{R}}$.*

Corollary 6.7.8. *Let $x > 0_{\mathbb{R}}$, and $n \in \mathbb{N}$, and $p \in \mathbb{Z}$ and $r \in \mathbb{Q}$. Then*

(i) $x^{n_{\mathbb{R}}} = x^n$,
(ii) *$x \neq 0_{\mathbb{R}}$ and $x^{p_{\mathbb{R}'}} = x^p$, and in particular, $x^{-1_{\mathbb{R}}} = x^{-1}$,*

(iii) $n \geq 1 \implies x \geq 0_{\mathbb{R}}$ *and* $x^{1_{\mathbb{R}}/n_{\mathbb{R}}} = x^{1/n}$, *and*
(iv) $x > 0_{\mathbb{R}}$ *and* $x^{r_{\mathbb{R}''}} = x^r$.

Proposition 6.7.9. *Let* $\alpha \in \mathbb{R}$. *Then* $(1_{\mathbb{R}})^{\alpha} = 1_{\mathbb{R}}$.

Proposition 6.7.10 (Properties of exponentiation of positive reals by reals)**.** *Let* $x, y > 0_{\mathbb{R}}$ *and* $\alpha, \beta \in \mathbb{R}$. *Then*

(i) $xy > 0_{\mathbb{R}}$ *and* $(xy)^{\alpha} = x^{\alpha}y^{\alpha}$,
(ii) $x^{\alpha+\beta} = x^{\alpha}x^{\beta}$,
(iii) $x^{-1} > 0_{\mathbb{R}}$, *and* $x^{\alpha} \neq 0_{\mathbb{R}}$ *and* $(x^{-1})^{\alpha} = (x^{\alpha})^{-1} = x^{-\alpha}$,
(iv) x^{α} *is a positive real, and*
(v) $|x| > 0_{\mathbb{R}}$ *and* $|x|^{\alpha} = |x^{\alpha}| = x^{\alpha}$.

Lemma 6.7.11. *Let* $x > 0_{\mathbb{R}}$ *and* $(a_i)_{i=m}^{\infty}$ *converge to* $0_{\mathbb{R}}$ *in* $\mathbb{R}$. *Then* $((x^a)_i)_{i=m}^{\infty}$ *converges to* $1_{\mathbb{R}}$ *in* $\mathbb{R}$.

Corollary 6.7.12 (Continuity of the real exponent)**.** *Let* $x > 0_{\mathbb{R}}$ *and* $(a_i)_{i=m}^{\infty}$ *be a convergent sequence of reals. Then* $((x^a)_i)_{i=m}^{\infty}$ *is a convergent sequence of reals with*

$$\lim_{n\to\infty} (x^a)_n = x^{\lim_{n\to\infty} a_n}.$$

Lemma 6.7.13. *Let* $x > 1_{\mathbb{R}}$ *and* $\alpha > 0_{\mathbb{R}}$. *Then* $x^{\alpha} > 1_{\mathbb{R}}$.

Proposition 6.7.14 (Further properties of exponentiation of positive reals by reals)**.** *Let* $x, y > 0_{\mathbb{R}}$ *and* $\alpha, \beta \in \mathbb{R}$. *Then*

(i) $\alpha > 0$ *and* $x > y > 0_{\mathbb{R}} \implies x^{\alpha} > y^{\alpha} > 0_{\mathbb{R}}$,
(ii) $x^{\alpha\beta} = (x^{\alpha})^{\beta}$,
(iii) *for each* $n \in \mathbb{N}$, *we have that* $x^n > 0_{\mathbb{R}}$ *and* $(x^n)^{\alpha} = (x^{\alpha})^n = x^(\alpha n_{\mathbb{R}})$,
(iv) *for each* $p \in \mathbb{Z}$, *we have that* $x^p > 0_{\mathbb{R}}$, *and* $x^{\alpha} \neq 0_{\mathbb{R}}$ *and* $(x^p)^{\alpha} = (x^{\alpha})^p = x^{\alpha p_{\mathbb{R}'}}$,
(v) *for each* $n \geq 1$, *we have that* $x^{1/n} > 0_{\mathbb{R}}$, *and* $x^{\alpha} \geq 0_{\mathbb{R}}$ *and* $(x^{1/n})^{\alpha} = (x^{\alpha})^{1/n} = x^{\alpha/n_{\mathbb{R}}}$, *and*
(vi) *for each* $r \in \mathbb{R}$, *we have that* $x^r > 0_{\mathbb{R}}$, *and* $x^{\alpha} > 0_{\mathbb{R}}$ *and* $(x^r)^{\alpha} = (x^{\alpha})^r = x^{\alpha r_{\mathbb{R}''}}$.

Corollary 6.7.15. *(i)* *Let* $x, y > 0_{\mathbb{R}}$ *and* $\alpha \in \mathbb{R}$ *such that* $x^{\alpha} = y^{\alpha}$. *Then* $x = y$.

(ii) *Let* $x \in \mathbb{R}$ *and* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha < \beta$. *Then*

(a) $0_{\mathbb{R}} < x < 1_{\mathbb{R}} \implies x^{\alpha} > x^{\beta}$, *and*

(b) $x > 1_{\mathbb{R}} \implies x^{\beta} > x^{\alpha}$.

Lemma 6.7.16 (Embeddings consistent with exponentiation)**.** *Let* $r \in \mathbb{Q}$, *and* $n \in \mathbb{N}$ *and* $p \in \mathbb{Z}$. *Then*

(i) $(r_{\mathbb{R}''})^n = (r^n)_{\mathbb{R}''}$, *and*
(ii) $r \neq 0_{\mathbb{Q}} \implies r_{\mathbb{R}''} \neq 0_{\mathbb{R}}$ *and* $(r_{\mathbb{R}''})^p = (r^p)_{\mathbb{R}''}$.