Contents

Chapter I

Set theory

1 Fundamentals

Declaration 1. Set is an object type and "$x \in A$" is a binary relation symbol obeying axiom of substitution.

Remark. *We'll shorten "X is an object of type* Set*" to "X is a set".*

Axiom 1. Let x, A be objects with $x \in A$. Then A is a set.

Axiom 2 (Extension)**.** Let A, B be sets such that $x \in A \iff x \in B$, for any object x. Then $A = B$.[1]

Axiom 3 (Axiom schema of replacement)**.** Let A be a set and P be a two-slot property obeying axiom of substitution such that for each object $x \in A$, there exists at most one object y such that $P(x, y)$ holds. Then there exists a (unique[2]) set $\{y : P(x, y) \text{ for some } x \in A\} =: X$ such that for any object y, we have that $y \in X \iff$ there exists an object $x \in A$ such that $P(x, y)$ holds.

Corollary 1.1 (Axiom schema of specification)**.** *Let A be a set and P be a one-slot property obeying axiom of substitution. Then there exists a unique set $\{x \in A : P(x)\} =: X$ such that $x \in X \iff x \in A$ and $P(x)$ holds, for any object x.*

Pseudo-axiom 4. There exists a set.[3]

[1] Converse of this is a property of equality.
[2] Due to Axiom 2.
[3] This will follow from Axiom 13.

Corollary 1.2 (Empty set)**.** *There exists a unique set* $\emptyset$ *that contains no object.*

Pseudo-axiom 5 (Pairing)**.** Let a, b be objects. Then there exists a set X such that $a \in X$ and $b \in X$.[4]

Corollary 1.3. *For any objects* a*,* b*, there exists a unique set* $\{a, b\}$ *which contains precisely* a *and* b*.*

Notation. *In case of a single object* a*, we'll use* $\{a\}$ *to stand for* $\{a, a\}$*.*

Axiom 6 (Unions)**.** Let $\mathcal{C}$ be a set each of whose elements are sets. Then there exists a set X such that $x \in A$ for some $A \in \mathcal{C} \implies x \in \mathcal{C}$, for any object x.

Corollary 1.4. *Let* $\mathcal{C}$ *be a set of sets. Then there exists a unique set* $\bigcup \mathcal{C}$ *such that* $x \in \bigcup \mathcal{C} \iff x \in A$ *for some* $A \in \mathcal{C}$*, for any object* x*.*

Corollary 1.5 (Intersections)**.** *Let* $\mathcal{C}$ *be a nonempty set of sets. Then there exists a unique set* $\bigcap \mathcal{C}$ *such that* $x \in \bigcap \mathcal{C} \iff x \in A$ *for each* $A \in \mathcal{C}$*, for any object* x*.*

Notation. *For sets* A *and* B*, we'll use* $A \cup B := \bigcup\{A, B\}$ *and* $A \cap B := \bigcap\{A, B\}$*.*

Corollary 1.6. *Let* a*,* b *be objects. Then*

$$\{a\} \cup \{b\} = \{a, b\}.$$

Corollary 1.7 (Pairwise unions and intersections)**.** *Let* A*,* B *be sets. Then for any object* x*, the following hold:*

(i) $x \in A \cup B \iff x \in A$ *or* $x \in B$*.*

(ii) $x \in A \cap B \iff x \in A$ *and* $x \in B$*.*

Definition 1.8 (Disjoint sets)**.** Sets A and B are called disjoint iff $A \cap B = \emptyset$.

Definition 1.9 (Subsets)**.** Let A, B be sets. Then we write "$A \subseteq B$" iff for any object x, we have that $x \in A \implies x \in B$.

Proposition 1.10. *Any set of sets is (weakly) partially ordered by* $\subseteq$*.*[5]

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[4] This will follow from Axiom 3 and the existence of any two element set, like $2^{2^{\emptyset}}$ which follows from Axiom 11.

[5] We haven't declared relations yet, let alone orders. But what matters here are the reflexive, antisymmetric, and transitive properties which can be stated without enunciating the concept of relations. Also see Theorem 6.2.

Notation (Difference of sets)**.** *For sets A, B, we set*

$$A \setminus B := \{x \in A : x \notin B\}.$$

Proposition 1.11 (Algebra of sets)**.** *Let A, B, C, X be sets. Then the following hold:*

$$\begin{aligned}
A \cup \emptyset &= A & A \cap \emptyset &= \emptyset & & \\
A \cup X &= X & A \cap X &= A & & \text{if } A \subseteq X \\
A \cup A &= A & A \cap A &= A & & \\
A \cup B &= B \cup A & A \cap B &= B \cap A & & \\
A \cup (B \cup C) &= (A \cup B) \cup C & A \cap (B \cap C) &= (A \cap B) \cap C & &
\end{aligned}$$

$$\begin{aligned}
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
X \setminus (A \cup B) &= (X \setminus A) \cap (X \setminus B) \\
X \setminus (A \cap B) &= (X \setminus A) \cup (X \setminus B)
\end{aligned}$$

Proposition 1.12. *Let A, B be sets. Then the following are equivalent:*

(i) $A \cup B = B$.

(ii) $A \subseteq B$.

(iii) $A \cap B = A$.

Proposition 1.13. *Let $\mathcal{C}$ be a nonempty set of sets and A, B be sets such that $A \subseteq X \subseteq B$ for each $X \in \mathcal{C}$. Then*

$$A \subseteq \bigcap \mathcal{C} \subseteq \bigcup \mathcal{C} \subseteq B.$$

Proposition 1.14 (Absorption laws)**.** *Let A, B be sets. Then*

$$A \cup (A \cap B) = A = A \cap (A \cup B).$$

Proposition 1.15 (Partitions)**.** *Let A, B, X be sets. Then the following are equivalent:*

(i) $\{A, B\}$ *is a partition of* X.

(ii) $A = X \setminus B$ *and* $B = X \setminus A$.

Also, $\{A \setminus B,\ A \cap B,\ B \setminus A\}$ *is a partition of* $A \cup B$.

2 Functions

Declaration 2. Func is an object type, and "$f\colon X \to Y$" and "$f\colon x \mapsto y$" are ternary relation symbols obeying axiom of substitution.[6]

Axiom 7 (Properties of the relation symbol "$f\colon X \to Y$")**.**

(i) Let f be a function. Then there exist objects X, Y such that $f\colon X \to Y$.

(ii) Let f, X, Y be objects such that $f\colon X \to Y$. Then the following hold:

  (a) f is a function and X, Y are sets.

  (b) $f\colon x \mapsto y \implies x \in X$ and $y \in Y$, for any objects x, y.

  (c) For each object $x \in X$, there exists a unique y such that $f\colon x \mapsto y$.

(iii) Let f, X, Y, Y' be objects such that $f\colon X \to Y$ and $f\colon X \to Y'$. Then $Y = Y'$.

Axiom 8 (Equality of functions)**.** Let f, g, X, Y be objects such that $f\colon X \to Y$ and $g\colon X \to Y$. For any objects x, y, y', let $f\colon x \mapsto y$ and $g\colon x \mapsto y' \implies y = y'$. Then $f = g$.

Axiom 9 (Functions via functional properties)**.** Let X and Y be sets, and P be a two-slot property obeying axiom of substitution such that for each $x \in X$, there exists a unique $y \in Y$ such that $P(x, y)$ holds. Then there exists a function f such that the following hold:

(i) $f\colon X \to Y$.

(ii) $f\colon x \mapsto y \implies P(x, y)$, for any objects x, y.

Definition 2.1 (Domains and codomains)**.** Let f be a function and X, Y be sets. Then X is called a domain of f iff $f\colon X \to Y'$ for some set Y. We also call Y a codomain iff $f\colon X' \to Y$ for some set X'.

Proposition 2.2. *Functions uniquely determine their domains and codomains.*

Notation. *For a function f and an object x in its domain, we'll denote by $f(x)$ or f_x, the unique y in its codomain for which $f\colon x \to y$.*

Remark. *"$f\colon X \to Y$ is a function" stands for the fact that f is a function and X, Y are sets such that $f\colon X \to Y$.*

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[6] See §3.

Proposition 2.3 (Equality of functions). *Let $f, g\colon X \to Y$ be functions. Then $f = g \iff f(x) = g(x)$ for each $x \in X$.*

Proposition 2.4 (Functions via functional properties). *Let X, Y be sets and P be a two-slot property obeying axiom of substitution such that for each $x \in X$, there exists a unique $y \in Y$ such that $P(x, y)$ holds. Then there exists a unique function $f\colon X \to Y$ such that $P(x, f(x))$ holds for each $x \in X$.*

Proposition 2.5 (Function compositions). *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. Then there exists a unique function $g \circ f\colon X \to Z$ such that*

$$(g \circ f)(x) = g(f(x)).$$

Proposition 2.6. *Function composition is associative.*

Definition 2.7 (Injections, surjections and bijections). Let $f\colon X \to Y$ be a function. Then f is called

(i) injective iff $f(x) = f(y) \implies x = y$.
(ii) surjective iff codomain of f is Y.
(iii) bijective iff f is injective and surjective, both.

Proposition 2.8. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. Then the following hold:*

(i) f, g are injective (respectively surjective) $\implies$ $g \circ f$ is injective (respectively surjective).
(ii) $g \circ f$ is injective $\implies$ f is injective.
(iii) $g \circ f$ is surjective $\implies$ g is surjective.

Proposition 2.9 (Cancellation). *Let $f, \tilde{f}\colon X \to Y$ and $g, \tilde{g}\colon Y \to Z$ be functions with f surjective and g injective. Then the following hold:*

(i) $g \circ f = g \circ \tilde{f} \implies f = \tilde{f}$.
(ii) $g \circ f = \tilde{g} \circ f \implies g = \tilde{g}$.

Proposition 2.10 (Inclusion functions). *Let X be a set and $Y \subseteq X$. Then there exists a unique function $\iota_{X \leftarrow Y}\colon Y \to X$ such that $\iota_{X \leftarrow Y}\colon y \mapsto y$ for each $y \in Y$.*

Notation (Identity functions). *For a set X, we set* $\mathrm{id}_X := \iota_{X \leftarrow X}$.

Corollary 2.11.

(i) For sets X, Y, Z such that $Z \subseteq Y \subseteq X$, we have

$$\iota_{Z\leftarrow Y} \circ \iota_{Y\leftarrow X} = \iota_{Z\leftarrow X}.$$

(ii) For a function $f\colon X \to Y$, we have

$$f \circ \iota_{X\leftarrow X} = f = \iota_{Y\leftarrow Y} \circ f.$$

Definition 2.12 (Invertible functions)**.** A function $f\colon X \to Y$ is called invertible iff there exists a function $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Corollary 2.13. *Inverse of a function, if existent, is unique.*

Proposition 2.14. *A function is invertible $\iff$ it is bijective.*

Proposition 2.15. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be invertible. Then f^{-1} and $g \circ f$ are also invertible with*

$$\begin{gathered}(f^{-1})^{-1} = f, \text{ and}\\ (g \circ f)^{-1} = f^{-1} \circ g^{-1}.\end{gathered}$$

Lemma 2.16 (Pasting functions)**.** *Let $f\colon X \to Z$ and $g\colon Y \to Z$ be functions such that they agree on $X \cap Y$. Then there exists a unique function $h\colon X \cup Y \to Z$ such that*

$$h(x) = \begin{cases} f(x), & x \in X \\ g(x), & x \in Y \end{cases}.$$

Definition 2.17 (Families)**.** Let f be a function with domain I. Then we write that "$\{f_\alpha\}_{\alpha\in I}$ is a family".

Proposition 2.18 (Generalized distributivity and De Morgan)**.** *Let X, A be sets and $\{B_\alpha\}_{\alpha\in I}$ be a nonempty family of sets. Then the following hold:*[7]

$$A \cup \left(\bigcap_{\alpha\in I} B_\alpha\right) = \bigcap_{\alpha\in I} A \cup B_\alpha \qquad X \setminus \left(\bigcup_{\alpha\in I} B_\alpha\right) = \bigcap_{\alpha\in I} X \setminus B_\alpha$$

$$A \cap \left(\bigcup_{\alpha\in I} B_\alpha\right) = \bigcup_{\alpha\in I} A \cap B_\alpha \qquad X \setminus \left(\bigcap_{\alpha\in I} B_\alpha\right) = \bigcup_{\alpha\in I} X \setminus B_\alpha.$$

[7] $\bigcap_{\alpha\in I} X \setminus B_\alpha$ is the set $\bigcap\{X \setminus B_\alpha : \alpha \in I\}$, which exists due to replacement. Similarly, others.

3 Images and inverse images

Definition 3.1 (Forward images)**.** Let $f\colon X \to Y$ be a function and $S \subseteq X$. Then we set[8]

$$f(S) := \{f(x) : x \in S\}.$$

Remark. *There is a possible collision: What if x is both, an element and a subset of the domain of f? We will be cautious.*

Corollary 3.2. *A function $f\colon X \to Y$ is surjective $\iff$ $f(X) = Y$.*

Lemma 3.3. *Let $f\colon X \to Y$ be a invertible and $U \subseteq Y$. Then*

$$f^{-1}(U) = \{x \in X : f(x) \in U\}.$$

Definition 3.4 (Inverse images)**.** Let $f\colon X \to Y$ be a function and $U \subseteq Y$. Then we set

$$f^{-1}(U) := \{x \in X : f(x) \in U\}.$$

Remark. *Lemma 3.3 guarantees no notational collision: Definition 3.4 extends the notation compatibly.*

Proposition 3.5. *Let $f\colon X \to Y$ be a function, $S, T \subseteq X$ and $U, V \subseteq Y$. Let $\{A_\alpha\}_{\alpha\in I}$ and $\{B_\beta\}_{\beta\in J}$ be families of subsets of X and Y respectively. Then the following hold:*[9]

$$f\Big(\bigcup_{\alpha\in I} A_\alpha\Big) = \bigcup_{\alpha\in I} f(A_\alpha) \qquad f^{-1}\Big(\bigcup_{\beta\in J} B_\beta\Big) = \bigcup_{\beta\in J} f^{-1}(B_\beta)$$

$$f\Big(\bigcap_{\alpha\in I} A_\alpha\Big) \subseteq \bigcap_{\alpha\in I} f(A_\alpha) \text{ if } I \neq \emptyset \qquad f^{-1}\Big(\bigcap_{\beta\in J} B_\beta\Big) = \bigcap_{\beta\in J} f^{-1}(B_\beta) \text{ if } J \neq \emptyset$$

$$f(S \setminus T) \supseteq f(S) \setminus f(T) \qquad f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

$$f^{-1}(f(S)) \supseteq S$$
$$f(f^{-1}(U)) \subseteq U$$

[8] The right-hand-side actually stands for the set $\{y : y = f(x) \text{ for some } x \in S\}$, which exists due to replacement. We'll omit such mentions later.

[9] $\bigcup_{\alpha\in I} f(A_\alpha)$ is the set $\bigcup\{f(A_\alpha) : \alpha \in I\}$, which exists due to replacement. Similarly, others.

Proposition 3.6. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. Let $S \subseteq X$ and $U \subseteq Z$. Then*

$$(g \circ f)(S) = g(f(S)), \text{ and}$$
$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)).$$

Proposition 3.7 (Characterizing injections and surjections via images)**.** *For a function f, the following hold:*

(i) f is injective $\iff$ $f^{-1}(f(S)) = S$ for any $S \subseteq X$.

(ii) f is surjective $\iff$ $f(f^{-1}(U)) = U$ for any $U \subseteq Y$.

4 Cartesian products

Declaration 3. OrdPair is an object type and "(x, y)" is a bivariate function symbol obeying axiom of substitution.[10]

Axiom 10 (Properties of the function symbol "(x, y)")**.**

(i) Let x, y be objects. Then (x, y) is an ordered pair.

(ii) Let x, y, x', y' be objects such that $(x, y) = (x', y')$. Then $x = x'$ and $y = y'$.

(iii) Let p be an ordered pair. Then there exist objects x, y such that $p = (x, y)$.

Corollary 4.1 (Coordinates of an ordered pair)**.** *Let p be an ordered pair. Then there exist unique objects x, y such that $p = (x, y)$.*

Proposition 4.2 (Pairwise Cartesian products)**.** *Let X, Y be sets. Then there exists a unique set $X \times Y$ which contains precisely the ordered pairs (x, y) such that $x \in X$ and $y \in Y$.*

Proposition 4.3. *Let A, C, D be sets, and $\{B_\alpha\}_{\alpha \in I}$ be a nonempty family of sets. Then the following hold:*

$$A \times \Big(\bigcup_{\alpha \in I} B_\alpha\Big) = \bigcup_{\alpha \in I}(A \times B_\alpha)$$

$$A \times \Big(\bigcap_{\alpha \in I} B_\alpha\Big) = \bigcap_{\alpha \in I}(A \times B_\alpha)$$

$$A \times (C \setminus D) = (A \times C) \setminus (A \times D)$$

There are corresponding statements for the other side.

[10]See §3.

Proposition 4.4. *Let A, B, C, D be sets. Then the following hold:*

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$$
$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$
$$(A \times B) \setminus (C \times D) \supseteq (A \setminus C) \times (B \setminus D)$$

Proposition 4.5 (Coordinate functions)**.** *Let X, Y be sets. Then there exist unique functions $\pi_X \colon X \times Y \to X$*[11] *and $\pi_Y \colon X \times Y \to Y$ such that for any $(x, y) \in X \times Y$, we have*[12]

$$\big(\pi_X(x, y),\ \pi_Y(x, y)\big) = (x, y).$$

Further, these coordinate functions are surjective.

Proposition 4.6 (Direct sum of two functions)**.** *Let $f \colon X \to Y$ and $g \colon X \to Z$ be functions. Then there exists a unique function $h \colon X \to Y \times Z$ such that*

$$h \colon x \mapsto (f(x), g(x)).$$

This function is also characterized by the fact that $h \colon X \to Y \times Z$, and

$$(\pi_Y \circ h, \pi_Z \circ h) = (f, g),$$

where π_Y, π_Z are coordination functions from $Y \times Z$ onto Y, Z.

Proposition 4.7. *Let I, J be nonempty sets and $\{A_{\alpha,\beta}\}_{(\alpha,\beta) \in I \times J}$ be a family of sets. Then the following hold:*[13]

$$\bigcup_{\alpha \in I} \Big(\bigcup_{\beta \in J} A_{\alpha,\beta} \Big) = \bigcup_{(\alpha,\beta) \in I \times J} A_{\alpha,\beta} = \bigcup_{\beta \in J} \Big(\bigcup_{\alpha \in I} A_{\alpha,\beta} \Big)$$

$$\bigcap_{\alpha \in I} \Big(\bigcap_{\beta \in J} A_{\alpha,\beta} \Big) = \bigcap_{(\alpha,\beta) \in I \times J} A_{\alpha,\beta} = \bigcap_{\beta \in J} \Big(\bigcap_{\alpha \in I} A_{\alpha,\beta} \Big)$$

Also,

$$\bigcup_{\alpha \in I} \Big(\bigcap_{\beta \in J} A_{\alpha,\beta} \Big) \subseteq \bigcap_{\beta \in J} \Big(\bigcup_{\alpha \in I} A_{\alpha,\beta} \Big).$$

Axiom 11 (Power sets)**.** Let A be a set. Then there exists a set $\mathcal{X}$ such that $S \subseteq A \implies S \in X$, for any object S.

[11] This notation is bad.
[12] Strictly, we should write $\pi_X((x, y))$.
[13] Strictly speaking, the left-hand-side is constructed via two instances of replacement.

Corollary 4.8. *Let A be a set. Then there exists a unique set 2^A which contains precisely the subsets of A.*

Definition 4.9 (Partial functions)**.** Let X, Y be sets. Then a partial function from X to Y is a function with a subset of X being its domain, and a subset of Y, its codomain.

Proposition 4.10 (Set of functions)**.** *The following are equivalent:*

(i) Axiom 11.

(ii) Let X, Y be sets. Then there exists a unique set Y^X containing precisely the functions from X to Y.

(iii) Let X, Y be sets. Then there exists a unique set containing all the partial functions from X to Y.

5 Natural numbers

Declaration 4. Nat is an object type, and we have a constant symbol 0, and a function symbol "x++" obeying axiom of substitution.

Axiom 12 (Peano axioms)**.**

(i) 0 is a natural number.

(ii) $n{+}{+}$ is a natural number for each natural number n.

(iii) $n{+}{+} \neq 0$ for any natural number n.

(iv) $m{+}{+} = n{+}{+} \implies m = n$ for any natural numbers m, n.

(v) Let P be a one-slot property such that $P(0)$ holds and $P(n)$ holds $\implies P(n{+}{+})$ holds for any natural number n. Then $P(n)$ holds for all natural numbers n.

Axiom 13 (Infinity)**.** There exists X such that n is a natural number $\implies n \in X$, for each object n.[14]

Corollary 5.1. *There exists a unique set $\mathbb{N}$ that contains precisely the natural numbers.*

Remark. *We'll use "$x_0, x_1, \ldots$ is a family" to mean that $\{x_i\}_{i\in\mathbb{N}}$ is a family.*

Notation. *We'll use $1 := 0{+}{+}$, etc.*

[14] See Proposition 7.6 for why this is called so.

Proposition 5.2 (Natural cuts)**.** *Let $N \in \mathbb{N}$. Then there exist unique sets $\{0, \dots, N\} =: A$, and B such that the following hold:*

(i) $\{A, B\}$ *is a partition of* $\mathbb{N}$.

(ii) $0 \in A$.

(iii) $N{+}{+} \in B$.

(iv) $n \in A \setminus \{N\} \implies n{+}{+} \in A$.

(v) $n \in B \implies n{+}{+} \in B$.

Further, we also have the following:

(i) $N \in \{0, \dots, N\}$.

(ii) $\{0, \dots, N{+}{+}\} = \{0, \dots, N\} \cup \{N{+}{+}\}$.

Proposition 5.3 (Recursion)**.** *Let X be a set and $c \in X$. Let $f_0, f_1, \dots$ be functions $X \to X$. Then there exists a unique function $g \colon \mathbb{N} \to X$ such that*

$$g(0) = c, \text{ and}$$
$$g(n{+}{+}) = f_n(g(n)) \text{ for } n \in \mathbb{N}.$$

Proposition 5.4 ("Uniqueness" of $\mathbb{N}$)**.** *Have primed versions of Declaration 4, Axiom 12, and Axiom 13, with $\mathbb{N}'$ containing all the primed naturals. Then there exists a unique function $f \colon \mathbb{N} \to \mathbb{N}'$ such that*

$$f(0) = 0', \text{ and}$$
$$f(n{+}{+}) = f(n){+}{+}' \text{ for each } n \in \mathbb{N}.$$

Further, any such f is a bijection.

Proposition 5.5 (Addition)**.** *Let $n \in \mathbb{N}$. Then there exists a unique function $\mathbb{N} \to \mathbb{N}$, denoted $m \mapsto m + n$, such that*

$$0 + n = n, \text{ and}$$
$$(m{+}{+}) + n = (m + n){+}{+} \text{ for } m \in \mathbb{N}.$$

Corollary 5.6. *For each $n \in \mathbb{N}$, we have*

$$n{+}{+} = n + 1.$$

Proposition 5.7 (Multiplication)**.** *Let $n \in \mathbb{N}$. Then there exists a unique function $\mathbb{N} \to \mathbb{N}$, denoted $m \mapsto mn$, such that*

$$0n = 0, \text{ and}$$
$$(m{+}{+})n = mn + n \text{ for } m \in \mathbb{N}.$$

Proposition 5.8 (Algebra in $\mathbb{N}$). *Let $a, b, c \in \mathbb{N}$. Then the following hold:*

$$\begin{aligned} a + b &= b + a, \\ (a + b) + c &= a + (b + c), \\ a + 0 &= a, \\ ab &= ba, \\ (ab)c &= a(bc), \\ a1 &= a, \\ a(b + c) &= ab + ac, \text{ and} \\ (a + b)c &= ac + bc. \end{aligned}$$

Proposition 5.9 (No zero addends or zero divisors). *Let $a, b \in \mathbb{N}$. Then the following hold:*

(i) $a + b = 0 \implies a = 0 = b$.

(ii) $ab = 0 \implies a = 0$ *or* $b = 0$.

Proposition 5.10 (Cancellation for addition). *Let $a, b, c \in \mathbb{N}$. Then*

$$a + c = b + c \implies a = b.$$

Definition 5.11 (Order and positivity). Let $m, n \in \mathbb{N}$. Then we write

(i) $m \leq n$ iff $n = m + p$ for some $p \in \mathbb{N}$;[15] and,

(ii) $m < n$ iff $m \leq n$ and $m \neq n$.

We also say that m is positive iff $m > 0$.

Proposition 5.12. *Let $m, n \in \mathbb{N}$. Then*

$$m < n \iff m + 1 \leq n.$$

Proposition 5.13. $\leq$ *is a total order on* $\mathbb{N}$.

Proposition 5.14 (Behaviour with addition and multiplication). *Let $a, b, c \in \mathbb{N}$. Then the following hold:*

(i) $a < b \implies a + c < b + c$.

(ii) $a < b$ *and* $c > 0 \implies ac < bc$.

[15] Clearly, $n \leq n$ and hence Theorem 6.2 applies.

Corollary 5.15 (Cancellation for multiplication)**.** *Let* $a, b, c \in \mathbb{N}$ *and* $c \neq 0$*. Then*

$$ac = bc \implies a = b.$$

Proposition 5.16 (Equivalent forms of induction)**.** *The following are equivalent.*

(i) *The usual induction as in Axiom 12.*

(ii) (Induction from nonzero base case)**.** *Let* $m_0 \in \mathbb{N}$ *and* P *be a one-slot property such that* $P(m_0)$ *is true, and* $P(m) \implies P(m+1)$ *for each* $m \geq m_0$*. Then* $P(m)$ *holds for each* $m \geq m_0$*.*

(iii) (Strong induction)**.** *Let* $m_0 \in \mathbb{N}$ *and* P *be a one-slot property such that for each* $m \geq m_0$*, if* $P(n)$ *holds for each* $m_0 \leq n < m$*, then* $P(m)$ *holds. Then* $P(m)$ *holds for each* $m \geq m_0$*.*

(iv) (Backwards induction)**.** *Let* $m_0 \in \mathbb{N}$ *and* P *be a one-slot property such that* $P(m_0)$ *holds, and* $P(m+1) \implies P(m)$ *for each* $m < m_0$*. Then* $P(m)$ *holds for each* $m \leq m_0$*.*

(v) (Principle of infinite descent)**.** *Let* P *be a one-slot property such that for every* $n \in \mathbb{N}$*, if* $P(n)$ *holds, then there exists an* $m < n$ *such that* $P(m)$ *holds. Then* $P(n)$ *is false for each* $n \in \mathbb{N}$*.*

(vi) (Well-ordering)**.** *Let* $S \subseteq \mathbb{N}$ *be nonempty. Then there exists a least element in* S*.*

Proposition 5.17 (Euclid's division lemma)**.** *Let* $m, n \in \mathbb{N}$ *such that* $n \neq 0$*. Then there exist unique* $q, r \in \mathbb{N}$ *such that* $0 \leq r < n$ *and*

$$m = nq + r.$$

Miscellany

Definition 5.18 (Exponentiation)**.** Let $n \in \mathbb{N}$. Then there exists a unique function $\mathbb{N} \to \mathbb{N}$, denoted $m \mapsto n^m$, such that

$$\begin{aligned} n^0 &= 1, \text{ and} \\ n^{m{+}{+}} &= n^m\, n \text{ for } m \in \mathbb{N}. \end{aligned}$$

Remark. *The properties of exponentiation are proven in Result 7.13.*

Definition 5.19 (Odd and even naturals)**.** An $n \in \mathbb{N}$ is called

(i) even iff $n = 2m$ for some $m \in \mathbb{N}$; and,

(ii) odd iff $n = 2m + 1$ for some $m \in \mathbb{N}$.

Result 5.20. Odds and Evens partition $\mathbb{N}$.

6 n-fold Cartesian products

Lemma 6.1. *Let $n \in \mathbb{N}$. Then we have*

$$\{0, \ldots, n\} = \{i \in \mathbb{N} : 0 \leq i \leq n\}.$$

Definition 6.2 (Intervals in $\mathbb{N}$)**.** Let $m, n \in \mathbb{N}$. Then we define

$$\{m, \ldots, n\} := \{i \in \mathbb{N} : m \leq i \leq n\}, \text{ and}$$
$$\{m, m+1, \ldots\} := \{i \in \mathbb{N} : i \geq m\}.$$

Proposition 6.3. *Let $m, n \in \mathbb{N}$. Then*

$$\{1, \ldots, m\} = \{1, \ldots, n\} \implies m = n.$$

Definition 6.4 (n-tuples)**.** Let $n \in \mathbb{N}$. Then an n-tuple is a function with domain $\{1, \ldots, n\}$.

Notation. *We will also use the "$(x_1, \ldots, x_n)$" notation for n-tuples.*

We denote the unique 0-tuple by "$()$".

Definition 6.5 (n-fold Cartesian products)**.** Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be sets. Then we define[16]

$$\prod_{i=1}^{n} X_i := \left\{ x \in \left(\bigcup_{i=1}^{n} X_i \right)^{\{1,\ldots,n\}} : x_i \in X_i \text{ for } 1 \leq i \leq n \right\}.$$

Corollary 6.6. *For a set X, and an $n \in \mathbb{N}$, we have*

$$X^{\{1,\ldots,n\}} = \prod_{i=1}^{n} X.$$

[16] If we had defined n-tuples to be surjections, then the modified $\prod_i X_i$ would have to be specified from the set of partial functions from $\{1, \ldots, n\}$ to $\bigcup_i X_i$.

Notation. *In this case, we denote $X^{\{1,\ldots,n\}}$ by X^n.*

Result 6.7 (Generalized recursion)**.** Let X be a set and $c \in X$. Let $f_0, f_1, \ldots$ be functions with $f_i \colon X^{i+1} \to X$. Then there exists a unique function $g \colon \mathbb{N} \to X$ such that

$$g(0) = f_0(c), \text{ and}$$
$$g(n+1) = f_n(g(0), \ldots, g(n)) \text{ for } n \in \mathbb{N}.$$

Proposition 6.8 (Finite choice)**.** *Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be sets. Then the following are equivalent:*

(i) $\prod_{i=1}^{n} X_i \neq \emptyset$.

(ii) Each X_i is nonempty.

7 Cardinality of sets

Definition 7.1 (Equal cardinalities)**.** Sets will be said to have equal cardinalities, or be equinumerous, iff there exists a bijection between them.

Proposition 7.2. *Being equinumerous is an equivalence relation on any set of sets.*

Example 7.3. $\mathbb{N}$, odds, evens are all equinumerous.

Definition 7.4 (Finite sets)**.** A set X is said to

(i) have n elements iff X and $\{1, \ldots, n\}$ are in bijection for some $n \in \mathbb{N}$; and,

(ii) be finite iff X has n elements for some $n \in \mathbb{N}$; and,

(iii) be infinite iff it is not finite.

Proposition 7.5. *Let X be a finite set. Then there exists a unique $|X|$ such that X has $|X|$ elements.*

Proposition 7.6. $\mathbb{N}$ *is infinite.*

Lemma 7.7. *Let X be a finite set and $a \notin X$. Then*

$$|X \cup \{a\}| = |X| + 1.$$

Proposition 7.8. *Let X, Y be finite sets. Then $X \cup Y$, $X \cap Y$ are also finite with*

$$|X \cup Y| + |X \cap Y| = |X| + |Y|.$$

Proposition 7.9 (Subsets of finite sets)**.** *Let X be a finite set and $Y \subsetneq X$. Then Y is finite and*

$$|Y| < |X|.$$

Proposition 7.10 (Images of finite sets)**.** *Let $f\colon X \to Y$ be a function with X finite. Then the following hold:*

(i) *$f(X)$ is finite.*

(ii) *f is an injection $\implies |f(X)| = |X|$.*

(iii) *f is not an injection $\implies |f(X)| < |X|$.*

Proposition 7.11. *Let X, Y be finite sets. Then $X \times Y$ and Y^X are finite with*

$$|X \times Y| = |X|\,|Y|, \text{ and}$$
$$|Y^X| = |Y|^{|X|}.$$

Proposition 7.12. *Let X, Y, Z be sets. Then the following pairs have equal cardinalities:*

(i) *$Z^{Y \cup X}$ and $Z^Y \times Z^X$, if $Y \cap X = \emptyset$.*

(ii) *$Z^{Y \times X}$ and $(Z^Y)^X$.*

(iii) *$(Z \times Y)^X$ and $Z^X \times Y^X$.*

Result 7.13 (Properties of natural exponentiation)**.** Let $a, b, c \in \mathbb{N}$. Then the following hold:

$$a^{b+c} = a^b\, a^c$$
$$a^{bc} = (a^b)^c$$
$$(ab)^c = a^c\, b^c$$

Proposition 7.14. *Let X, Y be sets. Then the following are equivalent:*[17]

(i) *There is an injection $X \to Y$.*

(ii) *If $X \neq \emptyset$, then there is a surjection $Y \to X$.*

Definition 7.15 (Comparing arbitrary cardinalities)**.** A set X is said to have cardinality less than that that of a set Y iff there exists an injection $X \to Y$.

[17] Going from second to first requires AC. See **??**

Proposition 7.16 (Consistency with finite sets)**.** *Let X, Y be finite sets. Then X has cardinality less than that of Y $\iff$ $|X| \leq |Y|$.*

Result 7.17 (Pigeonhole principle)**.** Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n$ be finite sets. Then the following hold:

(i) $|\bigcup_{i=1}^{n} A_i| > n \implies$ some $|A_i| > 1$.

(ii) $|\bigcup_{i=1}^{n} A_i| < n \implies$ each $|A_i| \leq 1$.

sec 3: 19, 20, exponentiation

Chapter II

The number systems

We start with integers, having already discussed naturals in §5.

1 The integers

1.1 Axiomatizing existence

Axiom 14 (Integers)**.** There exists a totally ordered integral domain[1] with the nonnegatives being well-ordered.[2]

Definition 1.1 (Sets of integers)**.** We call any ring as in Axiom 14, a set of integers. We also define positive and negative integers via comparison with zero.

Remark. *We'll not fuss about the abuse of notation when we use the same notation for addition multiplication, etc. for naturals, integers, etc.*

Lemma 1.2. *In a set of integers, there are no integers between* 0 *and* 1*, and hence we have*

$$m < n \iff m + 1 \leq n.$$

Proposition 1.3 (Inductive sets in $\mathbb{Z}$)**.** *The nonnegatives and nonpositives in a set of integers obey Peano axioms.*

[1] We can in fact weaken it to just a nonzero ring with identity, or even have Peano-like axioms and then define operations and order.

[2] For us, rings will be sets along with binary operations, not another object type.

Corollary 1.4 (Simple induction for $\mathbb{Z}$)**.** *Let $\mathbb{Z}$ be a set of integers and P be a one-slot property such that $P(0)$ holds, and $P(n) \implies P(n+1)$, for all $n \geq 0$, and $P(n) \implies P(n-1)$ for all $n \leq 0$. Then $P(n)$ holds for each $n \in \mathbb{Z}$.*

Corollary 1.5 (Simple recursion for $\mathbb{Z}$)**.** *Let $\mathbb{Z}$ be a set of integers, and $\mathbb{N}$ be the set of nonnegative integers. Let $f_n, g_n \colon X \to X$ be functions for $n \in \mathbb{N}$ and $c \in X$. Then there exists a unique function $h \colon \mathbb{Z} \to X$ such that*

$$\begin{aligned} h(0) &= c, \\ h(n+1) &= f_n(h(n)) \text{ for } n \geq 0, \text{ and} \\ h(n-1) &= g_n(h(n)) \text{ for } n \leq 0. \end{aligned}$$

Proposition 1.6 ("Uniqueness" of $\mathbb{Z}$)**.** *There exists a unique ring isomorphism between any two sets of integers, which further preserves order too.*

Proposition 1.7 (Embedding $\mathbb{N}$ into $\mathbb{Z}$)**.** *From a set of naturals[3] to a set of integers, there exists a unique injection that preserves addition and multiplication. Order also gets preserved as a byproduct.*

Notation. *For the rest of the notes, we'll fix a concrete such set, and denote it by $\mathbb{Z}$. Also, we'll identify $\mathbb{N}$ with the nonnegative integers.*

Proposition 1.8 (Euclid's lemma)**.** *Let $m, n \in \mathbb{Z}$ with $n \neq 0$. Then there exist unique $q, r \in \mathbb{Z}$ such that[4] $0 \leq r < |n|$, and*

$$m = nq + r.$$

Proposition 1.9 (Inductive sets in $\mathbb{Z}$)**.** *Let $S \subseteq Z$ such that $S \neq \emptyset, \mathbb{Z}$ and for each $i \in \mathbb{Z}$, we have*

$$i \in S \implies i + 1 \in S \text{ (respectively } i - 1 \in S\text{).}$$

Then S contains least (respectively greatest) element.

[3] By "a set of naturals", we mean a set along with an object "zero" and a "successor" function that together obey Peano axioms. Hence, $\mathbb{N}$ is *a* set of naturals.

[4] Absolute value is defined in the usual way.

1.2 Constructing integers from naturals

Proposition 1.10 (The equivalence relation). *The following defines an equivalence relation on* $\mathbb{N} \times \mathbb{N}$*:*

$$(a,b) \sim (c,d) \quad \textit{iff} \quad a+d = c+b.$$

Notation. *We'll call the equivalence classes as "differences", and use*

$$\begin{aligned} a \text{ --- } b &:= [(a,b)], \text{ and} \\ \mathscr{Z} &:= \{a \text{ --- } b : a,b \in \mathbb{N}\}. \end{aligned}$$

Theorem 1.11 (Ring structure of $\mathscr{Z}$). *We can define the following operations on* $\mathscr{Z}$*:*

$$\begin{aligned} (a \text{ --- } b) + (c \text{ --- } d) &:= (a+c) \text{ --- } (b+d) \\ (a \text{ --- } b)(c \text{ --- } d) &:= (ac+bd) \text{ --- } (ad+bc) \end{aligned}$$

Further, $\mathscr{Z}$ *forms an integral domain with these operations. We have:*

$$\begin{aligned} \textit{zero} &= 0 \text{ --- } 0 \\ -(a \text{ --- } b) &= b \text{ --- } a \\ \textit{identity} &= 1 \text{ --- } 0 \end{aligned}$$

Theorem 1.12 (Order on $\mathscr{Z}$). *The following is a well-defined total order on* $\mathscr{Z}$*:*

$$a \text{ --- } b \leq c \text{ --- } d \quad \textit{iff} \quad a+d \leq c+b.$$

Further, this order is compatible with the ring operations of $\mathscr{Z}$*, and well orders the set of nonnegative differences.*

2 The rationals

Definition 2.1 (Sets of rationals). A set of rationals is a smallest[5] totally ordered field. We also define positive and negative rationals via comparison with zero.

[5] That is, if K is any totally ordered field, then K contains an isomorphic image of the mentioned field.

Proposition 2.2 ($\mathbb{Z}$ inside $\mathbb{Q}$)**.** *Let $\mathbb{Q}$ be a set of rationals,*[6] *and $\mathbb{Z}$ be a set of integers. Then there exists a unique injective homomorphism $\phi\colon \mathbb{Z} \to \mathbb{Q}$. Further, ϕ preserves order, and*

$$\mathbb{Q} = \{\phi(m)\,\phi(n)^{-1} : m, n \in \mathbb{Z}, n \neq 0\}$$

Proposition 2.3 (Ordering the field of fractions)**.** *Let $\mathbb{Q}$ be the field of fractions of $\mathbb{Z}$. Then*[7]

$$a/b < c/d \quad \textit{iff} \quad ad < cb \quad \textit{for } b, d > 0$$

is a well-defined order on $\mathbb{Q}$ that makes it a totally ordered field.

Corollary 2.4. *There exists a set of rationals.*

Proposition 2.5 ("Uniqueness" of rationals)**.** *There is a unique injective homomorphism from any set of rationals to any totally ordered field which also preserves order.*

In particular, between any two sets of rationals, there exists a unique isomorphism, which further preserves order too.

Notation. *For the rest of the text, we'll fix a concrete set of rationals, $\mathbb{Q}$. We'll also identify integers with with their copy in $\mathbb{Q}$.*

Proposition 2.6. *Every rational can be written as a ratio of coprime integers.*

Proposition 2.7 (Gaps in rationals)**.** *There exists no rational whose square is 2.*

But for each rational $\varepsilon > 0$, there exists a rational $r \geq 0$ such that $r^2 < 2 < (r + \varepsilon)^2$.

3 The reals

Definition 3.1 (Reals)**.** An ordered complete field is called a set of reals.

Proposition 3.2 ("Uniqueness" of reals)**.** *Between any two sets of reals, there exists a unique isomorphism which also preserves order.*

Remark. *We'll fix a concrete such set $\mathbb{R}$, and continue the identification ritual.*

Proposition 3.3. *n-th (and hence rational) roots*[8] *of positive reals exist.*

Corollary 3.4. *Rationals violate the least-upper-bound property.*

[6] The first statement holds for any totally ordered field $\mathbb{Q}$.
[7] $a/b := [(a, b)]$.
[8] Cf. §5.

3.1 Constructing reals from rationals

Definition 3.5 (Rational-Cauchy sequences)**.** A sequence of rationals $(a_i)_{i=1}^{\infty}$ is called rational-Cauchy[9] iff for every rational $\varepsilon > 0$, there exists an integer $N \geq 1$ such that for all integers $i, j \geq N$, we have

$$|a_i - a_j| < \varepsilon.$$

Proposition 3.6 (The equivalence relation)**.** *The following defines an equivalence relation on the set of rational-Cauchy sequences starting from* 1*:*

$$(a_i)_{i=1}^{\infty} \sim (b_i)_{i=1}^{\infty} \quad \textit{iff} \quad \begin{array}{c} \textit{there exists an integer } N \geq 1 \textit{ such that} \\ |a_i - b_i| < \varepsilon \\ \textit{for each integer } i \geq N \end{array}$$

Remark. *We call the equivalence classes as "reals", and use*

$$\operatorname*{LIM}_{i\to\infty} a_i := [(a_i)_{i=1}^{\infty}], \textit{ and}$$
$$\mathscr{R} := \{\operatorname*{LIM}_{i\to\infty} a_i : (a_i)_{i=1}^{\infty} \textit{ is a Cauchy sequence of rationals}\}.$$

Theorem 3.7 (Field structure of $\mathscr{R}$)**.** *The following are well-defined operations on* $\mathscr{R}$*, which make it into a field:*[10]

$$\operatorname*{LIM}_{i\to\infty} a_i + \operatorname*{LIM}_{i\to\infty} b_i := \operatorname*{LIM}_{i\to\infty}(a_i + b_i)$$
$$(\operatorname*{LIM}_{i\to\infty} a_i)(\operatorname*{LIM}_{i\to\infty} b_i) := \operatorname*{LIM}_{i\to\infty}(a_i b_i)$$

Further, we have the following:[11]

$$\begin{aligned} \textit{zero} &= \operatorname*{LIM}_{i\to\infty} 0 \\ -(\operatorname*{LIM}_{i\to\infty} a_i) &= \operatorname*{LIM}_{i\to\infty}(-a_i) \\ \textit{identity} &= \operatorname*{LIM}_{i\to\infty} 1 \\ (\operatorname*{LIM}_{i\to\infty} a_i)^{-1} &= \operatorname*{LIM}_{i\to\infty} a_i^{-1} \quad \textit{if } \operatorname*{LIM}_{i\to\infty} a_i \neq \operatorname*{LIM}_{i\to\infty} 0 \textit{ and each } a_i \neq 0 \end{aligned}$$

[9] With no reals being previously defined, they are *a priori* different from Cauchy sequences.
[10] Implicitly is implied that the sequences on the right-hand-sides are rational-Cauchy.
[11] See Footnote 10.

Theorem 3.8 (Order on $\mathscr{R}$)**.** *The following is a well-defined total order on $\mathscr{R}$:*

$$\operatorname*{LIM}_{i\to\infty} a_i < \operatorname*{LIM}_{i\to\infty} b_i \quad \textit{iff} \quad \begin{array}{c} \textit{there exists an integer } N \geq 1 \\ \textit{and a rational } c > 0 \textit{ such that} \\ b_i - a_i > c \\ \textit{for each integer } i \geq N \end{array}$$

Further, this order is compatible with the field operations on $\mathscr{R}$ and has the least upper bound property.

3.2 Extended reals

Definition 3.9. We define extended reals to be the set $\mathbb{R} \cup \{+\infty, -\infty\}$ such that the following hold:

(i) $-\infty, +\infty \notin \mathbb{R}$ are distinct.

(ii) The order of $\mathbb{R}$ is extended to $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ as:
- (a) $-\infty < x < +\infty$ for all $x \in \mathbb{R}$.
- (b) $-\infty < +\infty$.

(iii) The negation on $\mathbb{R}$ is extended to $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ as:
- (a) $-(-\infty) := +\infty$.
- (b) $-(+\infty) := -\infty$.

Proposition 3.10 (Order and negation)**.**

(i) The extended reals are totally ordered.

(ii) Double negation is identity.

(iii) Negation reverses order.

Proposition 3.11 (l.u.b. property)**.** *Every subset of extended reals has a least upper bound and a greatest lower bound.*

Remark. *Unless stated otherwise, we'll be considering $-\infty$, $+\infty$ for bounds of subsets of $\mathbb{R}$.*

Chapter III

Limits of sequences

$\mathbb{R}$ forms a metric space with $d(x, y) = |x - y|$.

1 Convergence and limit laws

Definition 1.1 (Convergence)**.** A sequence of reals $(x_i)_{i=m}^{\infty}$ is said to converge to an $L \in \mathbb{R}$, denoted

$$x_i \to L,$$

iff for each real[1] $\varepsilon > 0$, there exists an integer $N \geq m$ such that for each integer $i \geq N$, we have

$$|x_i - L| < \varepsilon.$$

Proposition 1.2. *A real sequence converges to at most one point.*

Notation. *This allows to use* $\lim_{i\to\infty} a_i$ *notation. (Note that the starting index is already "in" a.)*

Definition 1.3 (Cauchy sequences of reals)**.** A sequence of reals $(x_i)_{i=m}^{\infty}$ is said to be Cauchy iff for each real $\varepsilon > 0$, there exists an integer $N \geq m$ such that for all integers $i, j \geq N$, we have

$$|x_i - x_j| < \varepsilon.$$

Proposition 1.4 (Characterizing boundedness)**.** *Let* $E \subseteq \mathbb{R}$*. Then* E *is bounded* $\iff$ *there exists a real* M *such that* $|x| \leq M$ *for each* $x \in E$*.*

[1] Or equivalently, rational. We'll not mention this again.

Definition 1.5 (Blindness to initial conditions)**.** A one-slot property P is said to be blind to initial conditions iff for any sequence $(x_i)_{i=m}^{\infty}$, and for any integers n, k, N with $n \geq m$ and $N \geq 0$, the following are equivalent:

(i) P holds for $(x_i)_{i=m}^{\infty}$.
(ii) P holds for $(x_{i+N})_{i=m}^{\infty}$.
(iii) P holds for $(x_i)_{i=n}^{\infty}$.
(iv) P holds for $(x_{i-k})_{i=m+k}^{\infty}$.

Notation. *This allows us to be imprecise about the starting index of the sequence from notations: For instance, in* $\lim_i 1/n$*, the sequence could begin with any positive integer.*

Proposition 1.6. *Convergence, Cauchy-ness and boundedness of real sequences are blind to initial conditions.*

Proposition 1.7. *For real sequences,*

$$\textit{convergence} \implies \textit{Cauchy-ness} \implies \textit{boundedness.}$$

Result 1.8 (Some standard limits)**.** In $\mathbb{R}$, as $i \to \infty$, we have the following limits:

$$\begin{aligned} 1/i &\to 0 && \\ x^i &\to 0 && \text{if for } |x| < 1 \\ x^{r_i} &\to 1 && \text{if } r_i \to 0 \text{ in } \mathbb{Q}, \text{ for } x > 0 \end{aligned}$$

Theorem 1.9 (Limit laws)**.** *Let $a_i \to L$ and $b_i \to M$ in $\mathbb{R}$. Then the following hold:*

$$\begin{aligned} a_i + b_i &\to L + M && \\ a_i\, b_i &\to LM && \\ -a_i &\to -L && \\ a_i^{-1} &\to L^{-1} && \textit{if } L \neq 0 \textit{ and each } a_i \textit{ is nonzero} \end{aligned}$$

$$\begin{aligned} \min(a_i, b_i) &\to \min(L, M) \\ \max(a_i, b_i) &\to \max(L, M) \\ |a_i| &\to |L| \end{aligned}$$

Let $n \in \mathbb{Z}$ and $r \in \mathbb{Q}$. Then we also have the following:

$$\begin{aligned} a_i^n &\to L^n && \text{if } n \geq 0 \\ a_i^n &\to L^n && \text{if each } a_i \text{ is nonzero and } L \neq 0 \\ a_i^r &\to L^r && \text{if each } a_i > 0 \text{ and } L > 0 \end{aligned}$$

Theorem 1.10 (Monotone convergence)**.** *A monotone sequence of reals is convergent if and only if it is bounded.*

Miscellany

Proposition 1.11. *Let (a_i) be a sequence of reals and $(a_{f(k)})$, $(a_{g(l)})$ be its subsequences such that the ranges of f, g cover the domain*[2] *of a. Then the following are equivalent:*

(i) $a_i \to L$ *in* $\mathbb{R}$.

(ii) $a_{f(k)} \to L$ *and* $a_{g(l)} \to L$ *in* $\mathbb{R}$.

2 lim sup, lim inf and limit points

Definition 2.1 (Limit points of sequences)**.** An $L \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ is said to be a limit point of a sequence $(x_i)_{i=m}^{\infty}$ of reals iff one of the following holds:

(i) $L = -\infty$ and (x_i) is unbounded below.

(ii) $L = +\infty$ and (x_i) is unbounded above.

(iii) $L \in \mathbb{R}$ and for each real $\varepsilon > 0$ and each integer $N \geq m$, there exists an integer $i \geq N$ such that

$$|x_i - L| < \varepsilon.$$

Definition 2.2 ($\liminf$ and $\limsup$)**.** Let $(x_i)_{i=m}^{\infty}$ be a real sequence. Then we define

$$\liminf_{i\to\infty} x_i := \sup_{N \geq m} \big(\inf_{i \geq N} x_i\big), \text{ and}$$

$$\limsup_{i\to\infty} x_i := \inf_{N \geq m} \big(\sup_{i \geq N} x_i\big)$$

Proposition 2.3. *Limit points,* $\limsup$ *and* $\liminf$*'s are blind to the initial conditions.*

[2] Or a final segment thereof.

Theorem 2.4. *Let (x_i) be a sequence of reals, $L^- := \liminf_{i\to\infty} x_i$ and $L^+ := \limsup_{i\to\infty} x_i$. Then the following hold:*

(i) *L^-, L^+ are limit points of (x_i).*

(ii) *If $L \in \mathbb{R}$ is a limit point of (x_i), then $L^- \leq L \leq L^+$.*

(iii) *Limit points of a sequence of limit points of (x_i) are limit points of (x_i).*

(iv) *If (x_i) is convergent, then its limit is its only limit point.*

(v) *Let $L \in \mathbb{R}$. Then $x_i \to L \iff L^- = L = L^+$.*

(vi) *If (y_i) is another sequence of reals such that each $x_i \leq y_i$, then*

$$\liminf_{i\to\infty} x_i \leq \liminf_{i\to\infty} y_i, \text{ and}$$

$$\limsup_{i\to\infty} x_i \leq \limsup_{i\to\infty} y_i.$$

Corollary 2.5 (Squeeze test)**.** *Let $x_i, z_i \to L$ in $\mathbb{R}$. Let $(y_i) \in \mathbb{R}$ such that*

$$\min(x_i, z_i) \leq y_i \leq \max(x_i, z_i)$$

for all i. Then $y_i \to L$ as well.

Corollary 2.6 (Zero test)**.** *In $\mathbb{R}$, we have*

$$a_i \to 0 \iff |a_i| \to 0.$$

Theorem 2.7 ($\mathbb{R}$ is complete)**.** *Every Cauchy sequence of reals is convergent.*

Proposition 2.8 (On subsequences)**.** *Let (x_i) be a sequence of reals, and $(x_{f(i)})$ be its subsequence. Then the following hold:*

(i) *If $x_i \to L$, then $x_{f(i)} \to L$ too.*

(ii) *L is a limit point of $(x_i) \iff (x_i)$ has a subsequence converging to L.*

Theorem 2.9 (Bolzano-Weierstraß)**.** *Every bounded real sequence has a convergent subsequence.*

3 Real exponentiation

Lemma 3.1. *Every real is the limit of some convergent sequence of rationals.*

Notation. *$\mathbb{R}^-$, $\mathbb{R}^+$ will stand for negative, and respectively positive reals.*

Proposition 3.2 (Defining real exponentiation)**.** *Let $x \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$. Then there exists a unique $x^\alpha \in \mathbb{R}^+$ such that for any rational sequence (r_i) such that $r_i \to \alpha$, we have*

$$x^{r_i} \to x^\alpha.$$

Remark. *There's no notational collision with the rational powers already defined for $\mathbb{R}^+$.*

Proposition 3.3 (Algebraic properties)**.** *Let $x, y, \alpha, \beta \in \mathbb{R}$ with $x, y > 0$. Then the following hold:*

$$\begin{aligned} x^{\alpha+\beta} &= x^\alpha\, x^\beta \\ x^{\alpha\beta} &= (x^\alpha)^\beta \\ (xy)^\alpha &= x^\alpha\, y^\alpha \end{aligned}$$

Proposition 3.4 (Order properties)**.**

(i) $t \mapsto t^\gamma$ is strictly increasing (respectively decreasing) if $\gamma > 0$ (respectively $\gamma < 1$).

(ii) $\gamma \mapsto t^\gamma$ is strictly increasing (respectively decreasing) if $t > 1$ (respectively $t < 1$).

Proposition 3.5 (Continuity)**.**

(i) Let $x_i \to L$ in $\mathbb{R}^+$ and $\alpha \in \mathbb{R}$. Then

$$x_i^\alpha \to L^\alpha.$$

(ii) Let $x > 0$ and $a_i \to \alpha$ in $\mathbb{R}$. Then

$$x^{a_i} \to x^\alpha.$$

Chapter IV

Series

1 Finite series

Definition 1.1 (Convergence of series)**.** Let $(a_i)_{i=m}^{\infty}$ be a sequence of reals and $m_0 \geq m$ be an integer. Define

$$S_N := \sum_{i=m_0}^{N} a_i$$

for each $N \geq m_0$. Then series $\sum_{i=m_0}^{\infty} a_i$ is said to be (conditionally) convergent iff (S_N) is convergent, in which case, we also write

$$\sum_{i=m_0}^{\infty} a_i := \lim_{N\to\infty} S_N.$$

If (S_N) is divergent, then the series $\sum_{i=m_0}^{\infty} a_i$ is said to be divergent.

We also say that $\sum_{i=m_0}^{\infty} a_i$ is absolutely convergent iff $\sum_{i=m_0}^{\infty} |a_i|$ is convergent.

Proposition 1.2 (Characterizing convergent series)**.** *Let* $(a_i)_{i=m}^{\infty} \in \mathbb{R}$*. Then* $\sum_{i=m}^{\infty} a_i$ *is convergent* $\iff$ *for each* $\varepsilon > 0$*, there exists an* $N \geq m$ *such that for all* $p, q \geq N$*, we have*

$$\left|\sum_{i=p}^{q} a_i\right| < \varepsilon.$$

Corollary 1.3 (Zero test)**.** *Let* $(a_i)_{i=m}^{\infty} \in \mathbb{R}$*. Then*

$$\sum_{i=m}^{\infty} a_i \text{ converges} \implies a_i \to 0.$$

Proposition 1.4. *Let* $(a_i)_{i=m}^{\infty} \in \mathbb{R}$ *with* $\sum_{i=m}^{\infty}$ *converging absolutely. Then* $\sum_{i=m}^{\infty} a_i$ *converges conditionally as well, with*

$$\left|\sum_{i=m}^{\infty} a_i\right| \leq \sum_{i=m}^{\infty} |a_i|.$$

Proposition 1.5 (Series algebra)**.** *Let* $(a_i)_{i=m}^{\infty}, (b_i)_{i=m}^{\infty} \in \mathbb{R}$ *with* $\sum_{i=m}^{\infty} a_i$ *and* $\sum_{i=m}^{\infty} b_i$ *being convergent. Then*[1] *the following hold:*

$$\begin{aligned} \sum_{i=m+k}^{\infty} a_{i-k} &= \sum_{i=m}^{\infty} a_i && \text{for any } k \in \mathbb{Z} \\ \sum_{i=n}^{\infty} a_i &= \sum_{i=m}^{\infty} a_i - \sum_{i=m}^{n-1} a_i && \text{for any integer } n \geq m \\ \sum_{i=m}^{\infty} (a_i + b_i) &= \sum_{i=m}^{\infty} a_i + \sum_{i=m}^{\infty} b_i \\ \sum_{i=m}^{\infty} c a_i &= c \sum_{i=m}^{\infty} a_i && \text{for any } c \in \mathbb{R} \end{aligned}$$

Proposition 1.6 (Alternating series)**.** *Let* $(a_i)_{i=m}^{\infty} \in \mathbb{R}^+ \cup \{0\}$ *be monotonically decreasing. Then*

$$\sum_{i=m}^{\infty} (-1)^i a_i \text{ is convergent } \iff a_i \to 0.$$

Proposition 1.7 (Telescoping series)**.** *Let* $(a_i)_{i=m}^{\infty} \in \mathbb{R}$ *with* $a_i \to L$*. Then*

$$\sum_{i=m}^{\infty} (a_i - a_{i+1}) = a_m - L.$$

[1] Implied is the fact all the said series converge.

Appendices

Appendix A

Basics

1 Regularity

Axiom 15 (Regularity)**.** Let A be a nonempty set. Then there exists an object $x \in A$ such that x is not a set, or x is a set with $x \cap A = \emptyset$.

Proposition 1.1 (No "$\in$-cycles" allowed!)**.** *Let $A_1, \ldots, A_n$ be sets for $n \geq 1$. Then th following is false:*

$$A_1 \in A_2 \cdots \in A_n \in A_1.$$

2 Russel's paradox

Only in this section will we use the following axiom. We will however continue to use the previous axioms.

Bad axiom (Unrestricted comprehension)**.** Let P be a one-slot property. Then there exists a set X such that $P(x)$ holds $\implies x \in X$, for any object x.

Corollary 2.1. *Let P be a one-slot property. Then there exists a unique set $\{x : P(x)\} =: X$ such that $x \in X \iff P(x)$ holds, for any object x.*

Corollary 2.2. *Axioms 3, 6, 11, 13, all become theorems.*

Corollary 2.3 (A contradiction!)**.** *Let*

$$X := \{x : x \notin x\}.$$

Then $X \in X \iff X \notin X$!

Corollary 2.4 (Collision with regularity!)**.** *Let*[1]

$$\Omega := \{x : x \textit{ is an object}\}.$$

Then $\Omega \in \Omega$*!*

3 Set theoretic formulation of different objects

Definition 3.1 (Making functions, ordered pairs)**.**

(i) For objects x, y, f, write "$f\colon x \mapsto y$" iff there exist objects $\not{f}$, Y such that $f = (\not{f}, Y)$ and $(x, y) \in \not{f}$.

(ii) For objects x, y, f, we write "$f\colon X \to Y$" iff the following hold:

   (a) X, Y are sets.

   (b) There exists an object $\not{f}$ such that $f = (\not{f}, Y)$ and $\not{f} \subseteq X \times Y$.

   (c) For each $x \in X$, there exists a unique object y such that $f\colon x \mapsto y$.

(iii) An object f is called a function iff there exist objects X, Y such that $f\colon X \to Y$.

Proposition 3.2. *Axioms 7 to 9, as well as the obedience of axiom of substitution in Declaration 2 become theorems.*

Definition 3.3 (Making ordered pairs, sets)**.**

(i) For objects x, y, we set $(x, y) := \{\{x\}, \{x, y\}\}$.[2]

(ii) An object p is called an ordered pair iff there exist objects x, y such that $p = (x, y)$.

Proposition 3.4. *Axiom 10 and the obedience of axiom of substitution in Declaration 3 become theorems.*

4 Finite products and sums

Definition 4.1 (Exponentiation)**.** Let S be a set with a binary operation, and $x \in S$. Then we define

$$x^1 := x, \text{ and}$$
$$x^{n+1} := x^n\, x \text{ for } x \geq 1.$$

[1] We could also have taken $\Omega = \{x : x \text{ is a set}\}$.

[2] Alternatively, we could also use $\{x, \{x, y\}\}$.

If S has an identity[3] e, we also define

$$x^0 := e,$$

and if the operation is associative and x invertible, then we also define

$$x^{-n} := (x^{-1})^n \text{ for } n \geq 1.$$

Remark. *There's no notational collision for x^{-1}.*

Proposition 4.2 (Properties of exponentiation)**.** *Let S be a set with an associative binary operation. Then the following hold:*

$$x^{mn} = (x^m)^n$$
$$x^{m+n} = x^m\, x^n$$

These hold for all $m, n \geq 1$. If S has an identity, then these hold for $m, n \geq 0$, and if x is invertible too, then true for $m, n \in \mathbb{Z}$. If the operation is further commutative, then the same statements hold for

$$(xy)^m = x^m\, y^m.$$

Definition 4.3 (General products)**.** Let X be a set with a binary operation having an identity.[4] Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in X$. Then we define:

For $k \in \mathbb{Z}$:

$$\prod_{i=1}^{k} a_i := \begin{cases} e, & k < 1 \\ \left(\prod_{i=1}^{k-1} a_i\right) a_k, & 1 \leq k \leq n \\ \prod_{i=1}^{n} a_n, & k > n \end{cases}$$

For $1 \leq \alpha \leq \beta \leq n$:

$$\prod_{i=\alpha}^{\beta} a_i := \prod_{i=1}^{\beta-\alpha+1} b_i$$

[3] There will be at most one identity.
[4] This identity will be unique.

where $b\colon \{1,\dots,\beta-\alpha+1\}\to X$ is defined by $b_i := a_{i+\alpha-1}$

For $\boldsymbol{k, l} \in \mathbb{Z}$:

$$\prod_{i=k}^{l} a_i := \begin{cases} e, & n<1, \text{ or } k>l, \text{ or } k>n, \text{ or } l<1 \\ \prod_{i=\max(1,k)}^{\min(l,n)} a_i \end{cases}$$

Remark. *In the above, there are no notational collisions.*

Theorem 4.4 (Splitting products)**.** *Let X have an associative binary operation with identity. Let $n \in \mathbb{N}$ and $a_1,\dots,a_n \in X$. Then for any integers $k-1 \le m \le l$, we have*

$$\prod_{i=k}^{l} a_i = \prod_{i=k}^{m} a_i \prod_{i=m+1}^{l} a_i.$$

Proposition 4.5 (Products over finite sets)**.** *Let X be a set with an associative and commutative binary operation with identity. Let S be a finite set and $a\colon S\to X$. Let $T\subseteq S$. Then there exists a unique $\prod_{t\in T} a_t \in X$ such that for any bijection $f\colon\{1,\dots,n\}\to T$ for $n\in\mathbb{N}$, we have*

$$\prod_{t\in T} a_t = \prod_{i=1}^{n} a_{f(i)}.$$

Theorem 4.6 (Substitution decompositions and Fubini)**.** *Let X be a set with associative and commutative binary operation with identity. Let S, T be finite sets. Then the following hold:*

(i) Let $f\colon S\to T$ be a bijection and $a\colon T\to X$ be a function. Then

$$\prod_{s\in S} a_{f(s)} = \prod_{t\in f(S)} a_t.$$

(ii) Let S, T be disjoint and $a\colon S\cup T\to X$. Then

$$\prod_{u\in S\cup T} a_u = \prod_{s\in S} a_s \prod_{t\in T} a_t.$$

(iii) Let $a, b\colon S\to X$. Then

$$\prod_{s\in S} a_s b_s = \prod_{s\in S} a_s \prod_{s\in S} b_s.$$

(iv) Let $a\colon S \times T \to X$. *Then*

$$\prod_{(s,t)\in S\times T} a_{s,t} = \prod_{s\in S}\Big(\prod_{t\in T} a_{s,t}\Big) = \prod_{t\in T}\Big(\prod_{s\in S} a_{s,t}\Big) = \prod_{(t,s)\in T\times S} a_{s,t}.$$

Lemma 4.7 (Binomial coefficients)**.** *Let* $n, r \in \mathbb{N}$ *with* $r \le n$. *Then the number of subsets of* $\{1, \dots, n\}$ *having* r *elements is*[5]

$$\frac{n!}{r!(n-r)!}.$$

Proposition 4.8 (Binomial theorem)**.** *Let* X *be a set with two binary operations:* $(a, b) \mapsto a + b, ab$. *Let both the operations be commutative and associative with multiplication having an identity and distributing over addition. Then for any* $n \in \mathbb{N}$, *we have*

$$(x+y)^n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!}\, x^i\, y^{n-i}.$$

5 *n*-th roots and rational exponents

Definition 5.1 (n-th roots)**.** Let G be a group and $x, y \in G$. Let $n \in \mathbb{Z} \setminus \{0\}$. Then we say that y is an n-th root[6] of x iff

$$y = x^n.$$

Notation. *If the roots are unique, then we'll use* $x^{1/n}$ *notation. Note that 1-th and* -1*-th roots always exist (they are unique too!) with* $x^{1/1} = x$ *and* $x^{1/-1} = x^{-1}$, *so no notational collision happens.*

Corollary 5.2 (Properties of n-th roots)**.** *Let* G *be a group and* $m, n \in \mathbb{Z} \setminus \{0\}$. *Then the following hold:*

(i) If G *is abelian, and* α, β *respectively are* m*-th roots of* x, y, *then* $\alpha\beta$ *is an* m*-th root of* xy.

(ii) If α *is an* mn*-th root of* x, *then* α^n *is an* m*-th root of* x.

[5] Implicitly is being stated that the the denominator divides the numerator.

[6] A sufficient condition for uniqueness (not existence!) of n-th roots is the injectivity of $x \mapsto x^n$ functions.

(iii) If α is an m-th root of x, then α^n is an m-th root of x^n.

If the roots are unique (and existent), then these facts become:

$$(xy)^{1/m} = x^{1/m}\, y^{1/m}$$
$$x^{1/mn} = (x^{1/m})^{1/n}$$
$$(x^n)^{1/m} = (x^{1/m})^n$$

Definition 5.3 (Rational exponentiation)**.** Let G be a group such that the maps $x \mapsto x^n$ are injective[7] for $n \in \mathbb{Z} \setminus \{0\}$. Let $p, q \in \mathbb{Z}$ with $q \neq 0$. Then a $y \in G$ is called a (p/q)-th power of an $x \in G$ iff y is a q-th root of x^p.

If existent, a rational power is unique.

Notation. *We denote this by $x^{p/q}$. There is no collision with x^n and $x^{1/n}$.*

Proposition 5.4 (Properties of rational powers)**.** *Let G be a group with injective $x \mapsto x^n$ maps for $n \in \mathbb{Z} \setminus \{0\}$. Then the following hold:*

(i) If r-th and s-th powers of x exist, then[8]

$$x^{r+s} = x^r\, x^s.$$

(ii) If r-th power of x exists, and s-th power of x^r exists, then

$$x^{rs} = (x^r)^s.$$

(iii) If G is abelian, and r-th powers of x, y exist, then

$$(xy)^r = x^r\, y^r.$$

6 Order

Definition 6.1 (Binary relations)**.** A binary relation on a set X is a subset of $X \times X$.

Remark. *We will use the usual "xRy" notation for relations.*

[7] Injectivity is required for this to be well-defined.
[8] Implicitly stated is the fact that $(r + s)$-th root of x exists, *etc.*

Theorem 6.2 (Strict and weak orders)**.** *Let X be a set and $\leq$ and $<$ be two binary relations on it. Then the following pairs of equivalent statements:*[9]

(i) *(a)* $a \leq b$ *and* $a \neq b \implies a < b$*; and* $a \leq a$.
  (b) $a < b$ *or* $a = b \implies a \leq b$*; and* $a \not< a$.

(ii) *If (both of) the above statements hold, then we also have the following pairs:*
  (a) *(1)* $a \leq b \leq c \implies a \leq c$*; and* $a \leq b \leq a \implies a = b$.
    (2) $a < b < c \implies a < c$.
  (b) *(1)* $a \leq b$ *or* $a \geq b$*; and* $a \leq b \leq a \implies a = b$.
    (2) *Exactly one of* $a < b$, $a = b$, $a > b$ *holds.*

Proposition 6.3 (Facts on l.u.b. and g.l.b.)**.** *Let R be a relation on a set X. Thenthe following hold:*

(i) *X satisfies the lest-upper-bound property $\iff$ is satisfies the greatest-lower-bound property.*[10]

(ii) *Let u_i's and l_i's respectively be l.u.b.'s and g.l.b.'s of subsets $S_i \subseteq X$. Then the following hold:*
  (a) *If u is a l.u.b. of u_i's, then u is a l.u.b. of $\cup_i S_i$.*
  (b) *If l is a g.l.b. of l_i's, then l is a g.l.b. of $\cap_i S_i$.*

(iii) *If R is anti-symmetric, then any subset of X has at most one l.u.b. and at most one g.l.b.*

6.1 Ordered groups

Definition 6.4 (Ordered groups)**.** A group together with a total order such that

$$a < b \implies ac < bc$$

is called a right-ordered group. Similarly, there are left-ordered and bi-ordered groups.

Remark. *Unless stated otherwise, an ordered group will be a right-ordered group.*

Example 6.5. Positive rationals form an ordered group, whereas nonzero rationals don't.

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[9] Statements separated by ";" are separate.

[10] l.u.b.'s and g.l.b.'s for this general R are defined in the same way as for partial orders.

Proposition 6.6 (Positive and negative cones)**.** *Let G be an ordered group and*

$$P := \{g \in G : g > e\}.$$

Then the following hold:

(i) $x < y \iff yx^{-1} \in P$.

(ii) $PP \subseteq P$.

(iii) $P^{-1} = \{g \in G : g < e\}$.

(iv) P, $\{e\}$, P^{-1} *partition* G.

Proposition 6.7 (Characterizing via positive cones)**.** *Let G be a group and $P \subseteq G$ such that $PP \subseteq P$, and P, $\{e\}$, P^{-1} partition G. Define*

$$x < y \quad \text{iff} \quad yx^{-1} \in P.$$

Then this order makes G an ordered group with

$$P = \{g \in G : g > e\}.$$

Proposition 6.8. *Let G be an ordered group.*[11] *Let $n \geq 0$ and $a_1, \dots, a_n, b_1, \dots, b_n \in X$ such that each $a_i \leq b_i$. Then we have*

$$\prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} b_i.$$

Proposition 6.9 (Monotonicity of x^t)**.** *In an ordered group, the following hold:*[12]

(i) $x \mapsto x^t$ *is strictly increasing (respectively decreasing) for positive (respectively negative) rational t.*

(ii) $t \mapsto x^t$ *is strictly increasing (respectively decreasing) for $x > e$ (respectively $x < e$).*

Proposition 6.10 (Supremum of product set)**.** *Let G be a bi-ordered group with least upper bound property. Let $A, B \subseteq G$. Then we have*

$$\sup(AB) = (\sup A)(\sup B).$$

[11] We don't need associativity or inverses—just a right-invariant order and an identity.

[12] Domains contain appropriate elements such that x^t is defined.

Definition 6.11 (Absolute value)**.** Let G be an ordered abelian group and $x \in G$. Then we define

$$|x| := \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}.$$

Proposition 6.12 (Properties of absolute value)**.** *In an ordered abelian group, the following hold:*

(i) $|x| \geq 0$ *with equality holding* $\iff x = 0$.

(ii) $|nx| = |n|\,|x|$ *for* $n \in \mathbb{Z}$.[13] *In particular,* $|-x| = |x|$.

(iii) $|x| < \alpha \iff -\alpha < x < \alpha$.

(iv) $x \in \{-|x|, |x|\}$.

(v) $\big||x| - |y|\big| \leq |x+y| \leq |x| + |y|$. *Further,* $\left|\sum_{s\in S} a_s\right| \leq \sum_{s\in S}|a_s|$ *for any finite set* S *with* a_s*'s being group elements.*

Proposition 6.13 (Interpretations of Nx in fields)**.** *If a field contains a copy of naturals, and hence integers and rationals, then* Nx *is the same, whether considered as integer times a group element, or as a product of two field elements.*

Proposition 6.14 (Additional property of absolute value in fields)**.** *In an ordered field, we also have*[14]

$$|xy| = |x|\,|y|.$$

Further, $|x^n| = |x|^n$ *for* $n \geq 0$*, and for* $n < 0$ *as well if* $x \neq 0$. *In particular,* $|x^{-1}| = |x|^{-1}$ *for* $x \neq 0$.

6.2 Ordered Archimedean fields

Proposition 6.15 (Characterizing Archimedean-ness)**.** *Let* F *be an ordered field. Then the following are equivalent:*

(i) $\mathbb{N}$ *(or equivalently,* $\mathbb{Q}$*) is unbounded in* F.

(ii) For any $x, \varepsilon > 0$ *in* F*, there exists an* $N \in \mathbb{N}$ *such that* $N\varepsilon > x$.

(iii) For any $x, \varepsilon > 1$ *in* F*, there exists an* $N \in \mathbb{N}$ *such that* $\varepsilon^N > x$.

Proposition 6.16 (Consequences)**.** *Let* F *be an ordered Archimedean field. Then the following hold:*

[13] $\mathbb{Z}$ is also an ordered abelian group.
[14] cf. (ii) of Proposition 6.12.

(i) (Floor and ceiling)**.** *Let x be a rational. Then we can define*

$$\lfloor x \rfloor := \max \{i \in \mathbb{Z} : i \leq x\}, \text{ and}$$
$$\lceil x \rceil := \min \{i \in \mathbb{Z} : i \geq x\}.$$

These are characterized by the fact that these are unique integers such that

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1, \text{ and}$$
$$\lceil x \rceil - 1 < x \leq \lceil x \rceil.$$

(ii) Between any two field elements exist rationals (and irrationals too, if F has any).

(iii) Any field element is surrounded by arbitrarily close rationals (and irrationals too, if F has any).