Contents

Chapter I

Topology

1 General

January 20, 2023

Lemma 1.1. *Convergence and limits of sequences are preserved while going across subspaces.*

Definition 1.2 (Bolzano-Weierstraß property)**.** A topological space X is said to have the Bolzano-Weierstraß property iff every sequence has a convergent subsequence.

Definition 1.3 (Domains and regions)**.** A *domain* is a nonempty, connected, open subset. A *region* is a subset that is contained in the closure of its interior.[1]

Lemma 1.4 (Closure and limit points)**.** *Let E be a subspace of X with $(x_i) \in E$ and $x \in X$. Then the following hold:*

(i) $x_i \to x \implies x \in \overline{E}$. *The converse*[2] *holds if X is metrizable.*

(ii) $x_i \to x$ *with x_i's being distinct* $\implies x \in \ell(E)$. *The converse*[3] *holds if X is metrizable.*

[1] That is, it is a domain plus some of its limit points.
[2] CC used.
[3] DCC used.

2 Limits and continuity

January 20, 2023

Definition 2.1 (Limits of functions)**.** Let X, Y be topological spaces and $f\colon S \to Y$ where $S \subseteq X$. Let $c \in X$ and $L \in Y$. Then we write

$$f(x) \to L \text{ as } x \to c$$

iff for every open neighborhood V of L, there exists an open neighborhood U of c such that

$$f(U \cap S \setminus \{c\}) \subseteq V.$$

Definition 2.2 (Continuity)**.** A function $f\colon X \to Y$ between topological spaces is said to be continuous at $c \in X$ iff for every open neighborhood V of $f(c)$, the set $f^{-1}(V)$ contains an open neighborhood of c.

Proposition 2.3. *A function $f\colon X \to Y$ is continuous $\iff$ for any $A \subseteq X$, we have*

$$f(\overline{A}) \subseteq \overline{f(A)}.$$

Lemma 2.4 (Relation between limits and continuity)**.** *A function $f\colon X \to Y$ between topological spaces is continuous at $c \in X$ $\iff$ $f(x) \to f(c)$ as $x \to c$.*

Lemma 2.5 (Restrictions and limits)**.** *Let X, Y be topological spaces and $f\colon S \to T$ where $S \subseteq X$ and $T \subseteq Y$. Let $A \subseteq S$ and $f(A) \subseteq B \subseteq Y$. Define $g\colon A \to B$ by $x \mapsto f(x)$. Then for $c \in X$ and $L \in T \cap B$, the following hold:*

(i) $f(x) \to L$ as $x \to c$ $\implies$ $g(x) \to L$ as $x \to c$ (A being seen as the subspace of X).

(ii) The converse holds if we have that $U_0 \cap S \subseteq A$ for some open neighborhood U_0 of c (in X).

Lemma 2.6 (Restrictions and continuity)**.** *Let $f\colon X \to Y$ between topological spaces where Y is a subspace of a space Y'. Let $S \subseteq X$ and $f(S) \subseteq T \subseteq Y'$. Define $g\colon S \to T$ by $x \mapsto f(x)$. Then for $c \in S$, the following hold:*

(i) f is continuous at c $\implies$ g is continuous at c.

(ii) The converse holds if $U_0 \subseteq S$ for some open neighborhood U_0 of c (in X).

Proposition 2.7 (Limits of compositions)**.** *Let* $f\colon E \to F$ *and* $g\colon F \to Z$ *where* E*,* F *are subspaces of* X*,* Y *respectively. Let* $f(x) \to L$ *as* $x \to c$ *and* $g(y) \to M$ *as* $y \to L$*. Also assume that* $M = g(L)$ *if* $L \in F$*. Then*

$$(g \circ f)(x) \to M \text{ as } x \to c.$$

Lemma 2.8 (Regularity)**.** *For a space* X*, the following are equivalent:*

(i) Any point and a closed set not containing it can be separated by disjoint open sets.

(ii) Any open neighborhood of a point contains the closure of an open set containing the point.

Theorem 2.9 (Extending a function continuously)**.** *Let* $f\colon E \to Y$ *where* E *is a subspace of a space* X*, and* Y *a regular space. Let* $S \subseteq \overline{E} \setminus E$ *and* $g\colon S \to Y$ *be such that* $f(x) \to g(s)$ *as* $x \to s$ *for each* $s \in S$*. Then the extended function* $E \cup S \to Y$ *is continuous.*

Lemma 2.10 (Denseness)**.** *A subset is dense* $\iff$ *it intersects with each nonempty open set.*

Proposition 2.11. *A continuous function into a Hausdorff space is determined by its values on a dense subset of the domain.*

3 Compactness

January 20, 2023

Definition 3.1 (Covers)**.** A set $\{U_\alpha\}$ of (open) subsets of a topological space is said to be an (open) cover of a subset $E \subseteq X$ iff $\bigcup_\alpha U_\alpha \supseteq E$.

Definition 3.2 (Compact sets)**.** A subset E of a topological space X is said to be compact in X iff every open cover of E has a finite subcover.

Lemma 3.3. *Compactness is preserved while going across subspaces.*

Remark. *This allows to drop "in* X*" from "*E *is compact in* X*".*

Proposition 3.4. *Closed subsets of compact spaces are compact.*

Proposition 3.5. *Continuous image of a compact set is compact.*

4 Connectedness and path-connectedness

January 20, 2023

Definition 4.1 (Connectedness)**.** A space is said to be connected iff it can't be partitioned into two nonempty open sets.

Definition 4.2 (Path connectedness)**.** A *path* from a point x to a point y in a space X is a continuous function $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

A space is called *path connected* iff any two points can be connected by a path.

Corollary 4.3. *Path connectivity of points induces a partition of the space.*

Definition 4.4 (Linear continua)**.** A linear continuum is a totally ordered set with least-upper-bound property[4] such that between any two points lies another point.

Theorem 4.5 (Linear continua are connected)**.** *Convex subsets of linear continua are connected.*

Proposition 4.6. *Path connectedness* $\implies$ *connectedness.*[5]

Proposition 4.7. *Continuous image of a (path-)connected set is (path-)connected.*

Proposition 4.8. *Closure of connected is connected.*[6]

Corollary 4.9 (Intermediate value)**.** *Let* $f\colon X \to Y$ *be continuous where* X *is connected and* Y *is under order topology. Let* r *lie between* $f(x)$ *and* $f(x)$ *for* $x, z \in X$*. Then there exists a* $y \in X$ *such that* $r = f(y)$*.*

Example 4.10 (Topologist's sine curve)**.** Define

$$S := \{(x, \sin(1/x)) : x > 0\} \subseteq \mathbb{R}^2.$$

Then $\overline{S}$ is connected, but not path-connected!

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[4] We mean that *bounded* nonempty sets have least upper bounds.
[5] Converse is *not* true. See Example 4.10.
[6] Not true for path-connected! See Example 4.10.

5 Pointwise convergence

January 27, 2023

Definition 5.1 (Pointwise convergence)**.** Let X be a set and Y be a topological space. Then a sequence of function (f_n) on $X \to Y$ is said to converge pointwise to a function $f\colon X \to Y$ iff for each $x \in X$, we have that

$$f_n(x) \to f(x) \text{ as } n \to \infty.$$

Corollary 5.2. *In a Hausdorff space, the pointwise limit, if existent, is unique.*

Chapter II

Metric spaces

January 12, 2023

1 General

January 23, 2023

Lemma 1.1. *The subspace topology on a subset of a metric space is the same as the topology induced by the restricted metric.*

Lemma 1.2. *Cauchy-ness of sequences is preserved while going across metric subspaces.*

Proposition 1.3. *Metric is continuous as both $X \times X \to \mathbb{R}$ as well as $X \to \mathbb{R}$.*

Lemma 1.4. *Closed subsets of a complete space are precisely its complete subsets.*

2 Compactness

Definition 2.1 (Totally bounded)**.** A subset E of a metric space X is said to be totally bounded in S iff for each $\varepsilon > 0$, finitely many balls of radius ε cover E.

Lemma 2.2. *(Total) boundedness is preserved while going across metric subspaces.*

Lemma 2.3. *Finite unions of bounded sets are bounded.*

Theorem 2.4 (Compactness in metric spaces)**.** *Let X be a metric space and $E \subseteq X$. Then the following are equivalent:*[1]

(i) *E is complete and totally bounded.*

(ii) *E has the Bolzano-Weierstraß property.*

(iii) *E is compact.*

Corollary 2.5 (Extreme value)**.** *A continuous function from a compact space X to $\mathbb{R}$ achieves its maximum and minimum over X.*

3 Limits of functions

Proposition 3.1 (Uniqueness of limits of functions)**.** *Let X be a topological space and Y be a metric space. Let $f\colon S \to Y$ where $S \subseteq X$. Then for a point $c \in X$, the following hold:*

(i) $c \in \ell(S) \implies f(x) \to L$ *as* $x \to c$ *for at most one* $L \in Y$.

(ii) $c \notin \ell(S) \implies f(x) \to L$ *as* $x \to c$ *for all* $L \in Y$.[2]

Notation. *Thus, for a metric space codomain, and for $c \in \ell(S)$, we denote the unique limit, if existent, by*

$$\lim_{x \to c} f(x).$$

4 Uniform convergence

January 27, 2023

Definition 4.1 (Uniform convergence)**.** Let X be a set and Y be a metric space. Then a sequence of functions (f_n) on $X \to Y$ is said to converge uniformly to a function $f\colon X \to Y$ iff for every $\varepsilon > 0$, there exists an N such that for all $n \geq N$ and for each $x \in X$, we have

$$d(f_n(x), f(x)) < \varepsilon.$$

[1] AC used.

[2] This also holds when Y is a general topological space.

Corollary 4.2. *Uniform limit is also a pointwise limit and hence unique.*

Lemma 4.3 (Cauchy criterion for uniform convergence)**.** *Let X be a set and Y be a metric space. Let $f\colon X \to Y$ be the pointwise limit of a sequence of functions (f_n) on $X \to Y$. Then the following are equivalent:*

(i) $f_n \to f$ *uniformly.*

(ii) *For each $\varepsilon > 0$, there exists an N such that for all $m, n \geq N$, we have $d(f_m(x), f_n(x)) < \varepsilon$ for all $x \in X$.*

Theorem 4.4 (Uniform limit preserves continuity)**.** *Let E be a subspace of a topological space X and Y be a metric space. Let (f_n) be a sequence of functions $E \to Y$ converging uniformly to $f\colon E \to Y$. Let $c \in X$ and for each n, let $f_n(x) \to L_n$ as $x \to c$. Let $\lim_{n\to\infty} L_n = L$. Then $f(x) \to L$ as $x \to c$.*

5 Miscellaneous

January 27, 2023

Theorem 5.1 (Contraction mapping)**.** *Let X be a nonempty complete metric space and $f\colon X \to X$ be such that there exists a $c \in [0, 1)$ such that*

$$d(f(x), f(y)) \leq cd(x, y).$$

Then there exists a unique fixed point of f.

Chapter III

Normed linear spaces

Convention. *V will denote a general normed linear space over $\mathbb{K}$, and W a general vector space over $\mathbb{K}$.*

Generic bases of V and W will respectively be denoted by $\mathcal{B}$ and $\mathcal{C}$.

Ω will denote an open set in V.

1 Elementary facts

January 12, 2023

Proposition 1.1. *On V, the following functions are continuous:*

(i) Addition: As both, $V \times V \to V$, and as $V \to V$ with a fixed addend vector.

(ii) Scalar multiplication: As $\mathbb{K} \times V \to V$, as $V \to V$ with a fixed scalar and as $\mathbb{K} \to V$ with a fixed vector.

(iii) Norm.

Definition 1.2 (Banach spaces)**.** A Banach space is a complete normed linear space.

Proposition 1.3. *Finite-dimensional normed linear spaces are Banach.*

2 Norms on product spaces

March 12, 2023

Proposition 2.1 (l_p-norms on $V_1 \times \cdots \times V_n$)**.** *For any $p \in [1, \infty)$, the following defines a norm on $V_1 \times \cdots \times V_n$:*

$$\|v\|_p := \Big(\sum_{i=1}^{n} \|v_i\|^p\Big)^{1/p}$$

We also have the following norm:

$$\|v\|_\infty := \max_{1 \leq i \leq n} \|v_i\|$$

Further, all these norms are equivalent via

$$\|v\|_\infty \leq \|v\|_p \leq n^{1/p} \|v\|_p$$

and generate the product topology.

Lemma 2.2.

(i) $\|w\| := \max_{\tilde{e} \in \mathcal{C}} |w_{\tilde{e}}|$ *defines a norm on W.*

(ii) If $f \colon V \to W$ is a vector space isomorphism, then $w \mapsto \|f^{-1}(w)\|$ defines a norm on W with respect to to which, f becomes an isometry.

Theorem 2.3. *Any two norms on a finite-dimensional vector space are equivalent.*

Corollary 2.4. *Any linear map from a finite-dimensional normed linear space to an arbitrary normed linear space is continuous.*

Corollary 2.5. *If $\dim V < \infty$ and $V \cong W$ as vector spaces, then for any norm on W, we also have that $V \cong W$ as normed linear spaces.*

Corollary 2.6. *Let $V = U_1 \oplus \cdots \oplus U_n$, where U_i's are subspaces of V. Consider $U_1 \times \cdots \times U_n$ under any of the (equivalent) l_p-norms. Then*

$$\dim V < \infty \implies U_1 \times \cdots \times U_n \cong V \text{ as normed linear spaces.}$$

Remark. This fails for infinite-dimensional V's even though $n < \infty$.[1] *(The statement for $n = \infty$ is obviously false for this would mean that any two norms on a space are equivalent.)*

However, for finite-dimensional spaces, this means that the "topology can be recovered just from the basis".

[1]See https://math.stackexchange.com/a/4620769/673223.

Corollary 2.7 (Convergence in finite dimensions)**.** *In a normed linear space with a basis* $(v_1, \dots, v_n)$*,*

$$\sum_{k=1}^{n} a_k^{(i)} v_k \xrightarrow{i} \sum_{k=1}^{n} a_k v_k \iff a_k^{(i)} \xrightarrow{i} a_k \text{ for all } k\text{'s.}$$

Corollary 2.8 (Limits of functions in finite dimensions)**.** *Let* $\dim V < \infty$ *and* $f \colon E \to V$ *where* E *is a subset of a topological space* X*. Let* f *decompose into* $f_e \colon E \to \mathbb{K}$ *for* $e \in \mathcal{B}$*. Let* $c \in X$ *and* $v \in V$*. Then*

$$f(x) \to v \text{ as } x \to c \iff f_e(x) \to v_e \text{ as } x \to c \text{ for each } e \in \mathcal{B}.$$

3 The space $\mathrm{B}(X, V)$

March 11, 2023

Proposition 3.1. W^X *is a vector space for any set* X*.*

Definition 3.2 (Bounded functions)**.** A function $f \colon X \to V$, where X is any set, is called bounded iff $\|f(x)\|$'s are bounded for $x \in X$.

We also define

$$\mathrm{B}(X, V) := \{\text{bounded functions } X \to V\}.$$

Remark. *Bounded-ness of multi-linear maps is something completely different! See Definition 4.2.*

Proposition 3.3 (A norm on $\mathrm{B}(X, V)$)**.** *For any set* X*, we have that* $\mathrm{B}(X, V)$ *is a linear subspace of* V^X*, and*

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|$$

defines a norm on $\mathrm{B}(X, V)$*.*

Further, the convergence in $\|\cdot\|_\infty$ *is the same as uniform convergence (for bounded functions).*

Theorem 3.4. *If* V *is Banach, then so is* $\mathrm{B}(X, V)$*.*

4 The space $\mathrm{B}\mathcal{L}(V_1, \dots, V_n; W)$

March 11, 2023

Convention. *In this section, V_i's will denote normed linear spaces over $\mathbb{K}$.*

Notation. *We'll use the $\mathcal{L}(\{V_i\}; W)$ notation for the set of multi-linear maps. The semi-colon ";" will differentiate the usual "linear" case or the "multi-linear" case for $V_1 \to W$. However, in that case, we conveniently have $\mathcal{L}(V_1, W) = \mathcal{L}(V_1; W)$.*

Corollary 4.1. *$\mathcal{L}(V_1, \dots, V_n; W)$ is a linear subspace of $W^{V_1 \times \cdots \times V_n}$.*

Definition 4.2 (Bounded multi-linear maps)**.** Let W also have a norm. Then a multi-linear map $f \in \mathcal{L}(V_1, \dots, V_n; W)$, for $n \geq 0$,[2] is called bounded iff it is bounded as a function on the product of discs $\|x_i\| \leq 1$, or equivalently, iff there exists an $M \geq 0$ such that

$$\|f(x_1, \dots, x_n)\| \leq M\|x_1\| \cdots \|x_n\| \qquad (x_i \in V_i).$$

We also define

$$\mathrm{B}\mathcal{L}(V_1, \dots, V_n; W) := \{\text{bounded multi-linear maps } V_1 \times \cdots \times V_n \to W\}.$$

Proposition 4.3 (A norm on $\mathrm{B}\mathcal{L}(V_1, \dots, V_n; W)$)**.** *Let W also come equipped with a norm. Then for any $n \geq 0$, we have that $\mathrm{B}\mathcal{L}(V_1, \dots, V_n; W)$ is a subspace of $\mathcal{L}(V_1, \dots, V_n; W)$ and that*

$$\begin{aligned} \|f\| &:= \left\| f|_{D_1 \times \cdots \times D_n} \right\|_\infty \\ &= \sup_{\|x_i\| \leq 1} \|f(x_1, \dots, x_n)\| \\ &= \sup_{\|x\|_\infty \leq 1} \|f(x)\| \end{aligned}$$

defines a norm on $\mathrm{B}\mathcal{L}(V_1, \dots, V_n; W)$, where D_i is the unit disc of the i-th space.

For $n \neq 0$, we also have

$$\|f\| = \sup_{x_i \neq 0} \frac{\|f(x_1, \dots, x_n)\|}{\|x_1\| \cdots \|x_n\|}.$$

[2] Note that for $n = 0$, the set $\mathcal{L}(V_1, \dots, V_n; W)$ is a singleton containing the empty map.

Corollary 4.4. *Let W, U be normed linear spaces. Let $f \in \mathrm{B}\mathcal{L}(V_1, \ldots, V_n; W)$ and $g \in \mathrm{B}\mathcal{L}(W; U)$. Then $g \circ f \in \mathrm{B}\mathcal{L}(V_1, \ldots, V_n; U)$ with*

$$\|g \circ f\| \leq \|g\| \, \|f\|.$$

Corollary 4.5. *Let W also have a norm and let $\phi \in \mathrm{B}\mathcal{L}(V; W)$ be invertible with $\phi^{-1} \in \mathrm{B}\mathcal{L}(V; W)$ as well.*[3] *Then*

$$\|\phi\|, \|\phi^{-1}\| \leq 1 \implies \phi \text{ is an isometry} \implies \|\phi\| = 1 = \|\phi^{-1}\|.$$

Remark. *Up till now, we didn't need any norm on $V_1 \times \cdots \times V_n$. Just the norms on V_i's were sufficient to define a norm on $\mathrm{B}\mathcal{L}(V_1, \ldots, V_n; W)$.*

Corollary 4.6. *Let X be a bounded subset*[4] *of $V_1 \times \cdots \times V_n$ and W be normed. Then the restrictions of bounded multi-linear maps $V_1 \times \cdots \times V_n \to W$ to $X \to W$ form a subspace of $\mathrm{B}(X, W)$.*

Proposition 4.7 (Characterizing continuity)**.** *Let W have a norm and $f \in \mathcal{L}(V_1, \ldots, V_n; W)$. Then the following are equivalent:*

(i) f is continuous.

(ii) f is continuous at 0.

(iii) f is bounded.

Theorem 4.8 (Canonical isomorphisms)**.** *Let W', U_i, U_i''s be vector spaces for $1 \leq i \leq n$ ($0 \leq m \leq n$) over $\mathbb{K}$ such that $U_i \cong U_i'$ for each i and $W \cong W'$ as vector spaces via α_i's and β. Let σ be a permutation on $\{1, \ldots, n\}$. Then we have the following vector space isomorphisms:*[5]

$$\begin{aligned} \mathcal{L}(U_1, \ldots, U_n; W) &\cong \mathcal{L}(U_1', \ldots, U_n'; W') \\ \mathcal{L}(U_1, \ldots, U_n; W) &\cong \mathcal{L}(U_{\sigma(1)}, \ldots, U_{\sigma(n)}; W) \\ \mathcal{L}\big(U_1, \ldots, U_m; \mathcal{L}(U_{m+1}, \ldots, U_n; W)\big) &\cong \mathcal{L}(U_1, \ldots, U_n; W) \end{aligned}$$

The isomorphisms can be described as follows: Let f (in left-hand-side) and g (in right-hand-side) be a corresponded pair of maps. Then we demand that

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[3] This will also hold if we replace "$\mathrm{B}\mathcal{L}$" by "B", and "$\|\cdot\|$" by "$\|\cdot\|_\infty$".

[4] "Boundedness" can be talked of only when we have a metric on $V_1 \times \cdots \times V_n$.

[5] The first is the "axiom of substitution".

(i) $\beta(f(u_1, \ldots, u_n)) = g(\alpha_1(u_1), \ldots, \alpha_n(u_n))$ *for the first;*

(ii) $f(u_1, \ldots, u_n) = g(u_{\sigma(1)}, \ldots, u_{\sigma(n)})$ *for the second; and,*

(iii) $f(u_1, \ldots, u_m)(u_{m+1}, \ldots, u_n) = g(u_1, \ldots, u_n)$ *for the third.*

In the above, if U_i, U_i''s, W' are normed with α_i's, β being normed linear space isomorphisms,[6] *then the above vector space isomorphisms can also be restricted by replacing "$\mathcal{L}$" by "$\mathrm{B}\mathcal{L}$", yielding normed linear space isomorphisms. In fact, the obtained isomorphisms are isometries*[7] *in the second and third cases. In the first case, it becomes an isometry if α_i's and β are isometries.*

Theorem 4.9. *If W is Banach, then so is $\mathrm{B}\mathcal{L}(V_1, \ldots, V_n; W)$.*

5 Polygonal connectedness

January 20, 2023

Definition 5.1 (Polygonal lines in vector spaces[8] over $\mathbb{K}$)**.** For $w_1, w_2 \in W$, we define the *line segment*[9]

$$[w_1; w_2] := \{tw_1 + (1-t)w_2 : t \in [0, 1]\}.$$

We define $(u; v)$, *etc.* in the obvious way.

For $w_1, \ldots, w_n \in W$, where $n \geq 0$, we call $\bigcup_{i=1}^{n-1} [w_i; w_{i+1}]$ a *polygonal line.*

A subset $E \subseteq W$ is called *polygonally connected* iff any two points in W can be polygonally connected within E.

Corollary 5.2. *Polygonal connectedness $\implies$ path connectedness in a normed linear space.*

Corollary 5.3. *Polygonal connectivity of points induces a partition of a vector space.*

Lemma 5.4 (Balls of normed linear space are convex)**.** *If x, y are contained in a ball, then the segment $[x; y]$ also lies in the ball.*

[6] A linear isomorphism that is also a homeomorphism.

[7] We adopt the usual "metric space definition" of an isometry. (Note that it may not be bijective.) Then it follows that for normed linear spaces, isometries also preserve norms.

[8] This definition does not require V to be normed.

[9] Reference: https://planetmath.org/linesegment.

Proposition 5.5. *A domain of a normed linear space is polygonally connected.*

Proposition 5.6. *Let V be an inner-product space and $\mathcal{B}$ be orthonormal. Then any two points of a domain can be connected by a polynomial line, lying completely inside the domain, with each segment of the line being along one of the directions in $\mathcal{B}$.*

Result 5.7 (Balls have segments in normed spaces)**.** Let $x_0 \in \Omega$ and $v \in V \setminus \{0\}$. Then there exists an $\varepsilon > 0$ such that for each $t \in B_\varepsilon^{\mathbb{K}}(0)$, we have

$$x_0 + tv \in \Omega.$$

Consequently, singletons are not open in a nonzero normed linear space, and hence $\Omega \subseteq \ell(\Omega)$.

Result 5.8 (Closure of balls in normed linear spaces)**.** In a normed linear space, we have

$$\overline{B_r(x)} = D_r(x).$$

6 Convergence in norm

January 27, 2023

Definition 6.1 (Convergence in norm)**.** The series $\sum_{i=0}^{\infty} v_i$ in V is said to converge in norm[10] iff $\sum_{i=0}^{\infty} \|v_i\|$ converges in $\mathbb{R}$.

Corollary 6.2. *Convergence-in-norm implies Cauchy-ness.*

Proposition 6.3 (Characterizing completeness for normed spaces)**.** *V is Banach $\iff$ convergence-in-norm implies convergence for series.*

Remark. *This justifies calling convergence-in-norm as absolute convergence if V is Banach.*

[10]Note that we are not defining convergence-in-norm for sequences, just for series. This is an instance where the difference in "type" is crucial.

Corollary 6.4 (Weierstraß M-test)**.** *Let X be a set and $(f_n) \in \mathrm{B}(X, V)$. Let $(M_n) \in \mathbb{R}$ such that $\|f_n\|_\infty \leq M_n$ and $\sum_{n=0}^{\infty} M_n$ is convergent. Then the following hold:*

(i) $\sum_{n=0}^{\infty} f_n$ converges in norm (of $\mathrm{B}(X, V)$).

(ii) The partial sums $\sum_{n=0}^{k} f_n$ converge uniformly to $\sum_{n=0}^{\infty} f_n$.

Proposition 6.5 (Root test)**.** *Let $(v_i) \in V$ and define $L \in [0, \infty) \cup \{\infty\}$ as*

$$L := \limsup_{i \to \infty} \|v_i\|^{1/i}.$$

Then the following hold:

(i) $L < 1 \implies \sum_{i=0}^{\infty} v_i$ converges in norm.

(ii) $L > 1 \implies \left(\left\| \sum_{i=0}^{n} v_i \right\|\right)_n$ is unbounded.

Proposition 6.6 (Ratio test)**.** *Let $(v_n) \in V \setminus \{0\}$ and define $L^+, L^- \in [0, \infty) \cup \{\infty\}$ as*

$$L^+ := \limsup_{i \to \infty} \frac{\|v_{i+1}\|}{\|v_i\|}, \text{ and}$$
$$L^- := \liminf_{i \to \infty} \frac{\|v_{i+1}\|}{\|v_i\|}.$$

Then the following hold:

(i) $L^+ < 1 \implies \sum_{i=0}^{\infty} v_i$ converges in norm.

(ii) There exists an N such that $\|v_{i+1}\| / \|v_i\| \geq 1$ for all $i \geq N \implies \sum_{i=0}^{\infty} v_i$ diverges.

(iii) $L^- > 1 \implies \left(\left\| \sum_{i=0}^{n} v_i \right\|\right)_n$ is unbounded.

Remark. *Root test is more powerful than the ratio test. See Proposition 1.4.*

Proposition 6.7. *Sum and scalar multiplication preserve convergence-in-norm.*

7 Normed algebras

January 20, 2023

Definition 7.1 (Normed algebras)**.** A normed algebra is a $\mathbb{K}$-algebra together which is also a normed vector space with the following additional property:

$$\|uv\| \leq \|u\| \, \|v\|.$$

Example 7.2.

(i) $\mathcal{L}(W;W)$ forms an associative $\mathbb{K}$-algebra.

(ii) $\mathrm{B}\mathcal{L}(V;V)$ is a subalgebra of the normed algebra $\mathcal{L}(V;V)$.

Convention. *In this section, A will stand for a normed algebra.*

Proposition 7.3 (Limits of products of functions)**.** *Let $f, g \colon E \to A$ where E is a subset of a topological space X. Let $c \in X$ and $u, v \in V$ such that $f(x) \to u$ and $g(x) \to v$ as $x \to c$. Then, as $x \to c$,*

$$f(x)g(x) \to uv.$$

Proposition 7.4 (Limits of multiplicative inverses of functions)**.** *Let $f \colon E \to A$ where E is a subset of a topological space X. Let $\|uv\| = \|u\| \, \|v\|$ hold in A. Let $f(x) \neq 0$ and $f(x)^{-1}$ existent for each $x \in E$. Let $c \in X$ and $u \in V \setminus \{0\}$ such that u^{-1} exists and $f(x) \to u$ as $x \to c$. Then, as $x \to c$,*

$$f(x)^{-1} \to u^{-1}.$$

Definition 7.5 (Cauchy product of series)**.** Let R be a ring and $\sum_{i=0}^{\infty} a_i$, $\sum_{i=0}^{\infty} b_i$ be formal series in R. Then their Cauchy product is defined to be the formal series $\sum_{i=0}^{\infty} c_i$ where c_i is given by

$$c_i := \sum_{j+k=i} a_j b_k.$$

Example 7.6 (Cauchy product doesn't preserve convergence!)**.** The Cauchy product of the convergent series[11]

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

with itself diverges in $\mathbb{R}$.

Theorem 7.7 (Convergent-in-norm $\times$ convergent $=$ convergent)**.** *Let $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ converge to A and B respectively in $\mathcal{A}$. Further assume that $\sum_{i=0}^{\infty} a_i$ is convergent in norm. Then their Cauchy product converges to AB.*[12]

Proposition 7.8. *Cauchy product preserves convergence in norm.*

Corollary 7.9 (Cauchy-Hadamard on power series)**.** *Consider the formal power series $\sum_{i=0}^{\infty} c_i(x-x_0)^i$ with $c_i, x_0 \in \mathcal{A}$. Define $L := \limsup_{i\geq 0} \|c_i\|^{1/i} \in [0,\infty)\cup\{\infty\}$ and the radius of convergence of the series to be*

$$R := \begin{cases} 0, & L = \infty \\ 1/L, & 0 < L < \infty \\ \infty, & L = 0 \end{cases}.$$

Then for any $x \in \mathcal{A}$, the following hold:

(i) $\|x-x_0\| < R \implies \sum_{i=0}^{\infty} c_i(x-x_0)^i$ converges in norm.

(ii) $\|x-x_0\| > R \implies \left(\left\|\sum_{i=0}^{n} c_i(x-x_0)^i\right\|\right)_n$ is unbounded.

Now, define[13] *$f\colon B_R(x_0) \to \mathcal{A}$ such that for each $x \in B_R(x_0)$, we have*

$$\sum_{i=0}^{\infty} c_i(x-x_0)^i = f(x).$$

Then for any $\varepsilon > 0$, we have the uniform convergence

$$\sum_{i=0}^{n} c_i(x-x_0)^i \xrightarrow{n} f \text{ in } D_{R-\varepsilon}(x_0).$$

Also, f is continuous.

[11] Convergence follows from Corollary 7.11.

[12] No commutativity or associativity required!

[13] Obvious definitions of balls when $R = 0$ or ∞.

Remark.

(i) *Behaviour on the boundary of is not determined: In $\mathbb{C}$, while $\sum_{n=0}^{\infty} z^n$ converges nowhere on the boundary, $\sum_{n=0}^{\infty} z^n/n^2$ converges on every point of the boundary. Also see Corollary 7.12.*

(ii) *A power series that does not converge uniformly over it's ball of convergence: $\sum_{n=0}^{\infty} x^n$ in $\mathbb{R}$.*

7.1 Miscellaneous

January 27, 2023

Theorem 7.10. *Let $(v_i) \in V$ such that $\left(\left\| \sum_{i=0}^{n} v_i \right\|\right)_n$ is bounded. Let $a_0 \geq a_1 \geq \cdots \geq 0$ be reals with $\lim_{i \to \infty} a_i = 0$. Then the partial sums $\sum_{i=0}^{n} a_i v_i$ form a Cauchy sequence.*

Corollary 7.11 (Alternating series)**.** *Let $a_0 \geq a_1 \geq \cdots \geq 0$ be reals with $\lim_{i \to \infty} a_i = 0$. Then $\sum_{i=0}^{\infty} (-1)^i a_i$ is convergent.*

Corollary 7.12. *Let A be a normed algebra and $\sum_{i=0}^{\infty} a_i (x - x_0)^i$ be a power series with reals $a_0 \geq a_1 \geq \cdots \geq 0$. Then the series converges for $x \in B_1(x_0)$. We also have convergence on $D_1(x_0) \setminus \{x_0 + 1\}$ if the norm is multiplicative, or if the nonzeroes in A are invertible.*

8 Banach algebras

March 11, 2023

Definition 8.1 (Banach algebra)**.** A Banach space that also forms a $\mathbb{K}$ algebra, or equivalently, a complete normed linear space.

Example 8.2. If V is Banach, then $\mathrm{B}\mathcal{L}(V; V)$ is a Banach algebra.

Convention. *In this subsection, we'll take $\mathscr{B}$ to be a Banach algebra with identity,[14] and will set $\operatorname{Inv} \mathscr{B}$ to be the set of all the invertible elements of $\mathscr{B}$.*

[14] For we'll want our power series to start from 1.

Proposition 8.3 (Exponentiation)**.** *Let $x \in \mathscr{B}$. Then the series*

$$\sum_{i=1}^{\infty} \frac{1}{n!} x^n$$

converges in $\mathscr{B}$.

Notation. *We denote this sum by* $\exp(x)$, *or* e^x.

Proposition 8.4. *If x, y commute in $\mathscr{B}$, then*

$$e^x\, e^y = e^{x+y} = e^y\, e^x.$$

Corollary 8.5. $\exp\colon (\mathscr{B}, +) \to (\operatorname{Inv}\mathscr{B}, \cdot)$ *is a group homomorphism.*

Proposition 8.6. *If $\|x\| < 1$ in $\mathscr{B}$, then $1 - x$ is invertible with*

$$(1-x)^{-1} = \sum_{i=0}^{\infty} x^n$$

where right-hand-side is convergent.

Proposition 8.7 ($\operatorname{Inv}\mathscr{B}$ is open in $\mathscr{B}$)**.** *Whenever $x \in \operatorname{Inv}\mathscr{B}$, we have that $B_{1/\|x^{-1}\|}(x) \subseteq \operatorname{Inv}\mathscr{B}$.*

Theorem 8.8. *$x \mapsto x^{-1}$ is a self-homeomorphism on $\operatorname{Inv}\mathscr{B}$.*

Chapter IV

Reals and complex numbers

1 Reals

February 11, 2023

Proposition 1.1 (Characterizing lim sup)**.** *Let $(a_n) \in \mathbb{R}$ be bounded. Then $\limsup_{i\to\infty} a_i$ is the unique real L such that the following hold:*

(i) For every $\varepsilon > 0$, there exists an N such that $a_n < L + \varepsilon$ for all $n \geq N$.

(ii) For every $\varepsilon > 0$ and every N, there exists an $n \geq N$ such that $a_n > L - \varepsilon$.

Lemma 1.2. *Let $(a_n) \in (0, \infty)$ and $(b_n) \in [0, \infty)$ with $a_n \to a$ where $a \in (0, \infty) \cup \{\infty\}$. Then*[1]

$$\limsup_{i\to\infty} a_i b_i = a\big(\limsup_{i\to\infty} b_i\big).$$

Proposition 1.3. *Uniform limit of Riemann-integrable functions $[a, b] \to \mathbb{R}$ is Riemann-integrable.*

Proposition 1.4 (Root test better than ratio test)**.** *Let $(a_n) \in (0, \infty)$. Then*

$$\liminf_{i\to\infty} \frac{a_{i+1}}{a_i} \leq \liminf_{i\to\infty} a_i^{1/i} \leq \limsup_{i\to\infty} a_i^{1/i} \leq \limsup_{i\to\infty} \frac{a_{i+1}}{a_i}.$$

Remark.

[1] We are (perversely) allowing multiplication by ∞.

(i) *We can have* $\limsup a_i^{1/i} < 1 < \limsup a_{i+1}/a_i$*, or* $\liminf a_{i+1}/a_i < 1 < \limsup a_i^{1/i}$ *in which case, the ratio test gives no information, but root test does.*

(ii) *Since* $\liminf a_{i+1}/a_i \leq \limsup a_i^{1/i} \leq \limsup a_{i+1}/a_i$*, the root test gives answer whenever ratio test does!*